Applying Interacting Multiple Model to Financial Asset Allocation

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This paper describes a continuous-time state-process, discrete-time observation, Interacting Multiple Model (IMM) tracking algorithm, and its applications to financial market modeling and asset allocation. The system state is modeled as a continuous-time, affine-Gaussian stochastic dynamical process driven by a white process noise, as well as by structural changes modeled by a finite-state, continuous-time, Markov process. The system generally assumes multiple models with different state space dimensions, and an affine-Gaussian state jump whenever a model transition occurs. The underlying problem is a standard filtering problem for estimating the system state based on a sequence of discrete-time, linear-Gaussian observations of partial system states. To demonstrate the new method, we apply the IMM algorithm to financial market modeling for dynamic asset allocation. The resulting performance shows the potential applicability of the proposed method.

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1. INTRODUCTION

In this paper, we are generally concerned with financial market modeling and asset allocation problems, and specifically, with the possibility of applying Interacting Multiple Model (IMM) methods (which were developed as algorithms for tracking maneuvering targets [1] in 1980s, and since then, have been refined in many directions [3]) to financial market modeling. This paper expands the continuous-time IMM extrapolation algorithm introduced in [14] (which used a typical maneuvering target tracking example with stop-and-go target behavior as an illustration) to a full IMM tracking algorithm definition, and shows how the algorithm can be used to model financial market behaviors, as a continuous-time stochastic dynamical system with discrete-time observations, in which the system structure switches between multiple models.

Since the time when the IMM approach to tracking maneuvering targets was first published ([4, 5, 21]), the IMM methods have been widely used to make tracking algorithms adaptive to a wide range of target maneuvering and other abrupt structural changes in target motion dynamics. In fact, the IMM methods are one of the most studied subjects in target tracking, as documented in [3–7]. As a target tracking algorithm, each model used in an IMM algorithm typically represents a standard target behavior such as an almost-constant-velocity (called "nearly-constant-velocity" in [1]) model, and an almost-constant-rate turn model, or alternatively, multiple models may represent different levels of white process noises in the target dynamics so as to expand the range of tracking (filtering) bandwidth adaptively ([22]).

In a typical IMM implementation, both model switching and state transition are allowed to happen only on prescribed discrete time steps. Indeed, almost all the IMM literature starts with a discrete-time target dynamics formalism. As mentioned in [7], a few exceptions include [8, 24] in which the target dynamics are described by stochastic differential equations driven by Poisson processes (to model inter-model switching) as well as Wiener processes (to model intra-model diffusion). Those models are known as continuous-time Markov jump processes [2]. In contrast, the mathematical model used in this paper (first introduced in [14]) is expressed by a continuous-time Markov process on a hybrid state space explicitly through a semi-group of state transition operator and its infinitesimal generator. Like the model described in [9], our model allows switching across spaces with different dimensions, and as in [7, 10], our model allows the system state to jump whenever a model switching happens. These flexibilities have motivated us to explore the possibilities of applications to modeling of financial markets that exhibit similar behavior. As expressed in [17], our general motivation is to explore possibilities of applications of engineering techniques to social and economic system analysis.

Recently, switching models have been proposed to analyze financial markets, as described in [18–20]. The application of IMM methods for such modeling is very natural and apparently straightforward. The use of the continuous time IMM may be appropriate because, for example, the stock prices change almost constantly during the day but many people only pay attention to the closing prices (when other detailed data also become unavailable). The multiple model approach, as shown in [18–20], typically uses two models, i.e., bull (up) and bear (down) models. The continuous-time IMM algorithm shown in this paper allows us to switch among the models with different dimensions. For that reason, we will use three models where the third model, "steady" model, has different (reduced) dimension, and test the applicability of our new IMM algorithm to a more flexible financial market model.

The purpose of this paper is to investigate the applicability of our continuous time IMM algorithm to financial market. We will demonstrate its performance with a popular benchmark equity market index (S&P 500 futures) on different time scales. While we believe this model applies to general market dynamics, its overall effectiveness is subject to additional future research and validation.

In the next section, Section 2, we will define a continuous-time jump Markov linear/affine system as a Markovian process on a hybrid state space, expressed as a formal direct sum of Euclidean spaces with generally different dimensions. We then define a filtering problem, a solution to which is given in Section 3, where an IMM algorithm, with continuous time extrapolation and discrete time updating, will be described. Section 4 shows a simple three-model financial market model with an IMM extrapolation algorithm. Numerical examples of financial market modeling and asset allocation analysis will be presented in Section 5, followed by the conclusions in Section 6.

The preliminary version of this paper was presented at the 18th International Conference on Information Fusion [23].[1] We have refined the conference paper and added derivations of the approximation-less calculation of the model probability and the first and second moments of the state probability distribution for each model for the extrapolation step, a major technical contribution of the paper outlined in [8, 14].

2. JUMP MARKOV MODEL

Consider M models, each of which is represented by a vector-matrix triple (A_m, b_m, B_m) that defines an Itô's linear or affine stochastic differential equation as $dx_t = (A_m x_t + b_m)dt + B_m dw_t$, $m \in \{1,\ldots,M\}$, which defines a continuous-time stochastic process x_t on a Euclidean space E_m, with a vector-valued, unit-intensity Wiener process w_t, on an appropriate time interval. Thus, within a model m, the state x_t is a Gaussian stochastic process such that each sample is continuous (no jump).

We assume that model transition is expressed by a continuous-time, $\{1,\ldots,M\}$-valued, time-homogeneous Markov process $(m_t)_{t\in[t_0,\infty)}$ with transition probability

$$P_h(m' \mid m) \stackrel{\text{def}}{=} \text{Prob}\{m_{t+h} = m' \mid m_t = m\} = \begin{cases} c_{mm'}h + o(h) & \text{if } m' \neq m \\ 1 - \sum_{\substack{m'=1 \\ m' \neq m}}^{M} c_{mm'}h + o(h) & \text{otherwise} \end{cases} \tag{1}$$

for each $(m, m') \in \{1,\ldots,M\}^2$, $h > 0$, and $t \in [t_0, \infty)$, with constants $c_{mm'} \geq 0$ for $m' \neq m$, $c_{mm} = -\sum_{m'=1, m'\neq m}^{M} \cdot c_{mm'} < 0$, and a fixed initial time t_0. We assume each model transition is accompanied by an affine-Gaussian jump. Namely, when a model transition from m to m' happens at time t, the target state jumps from $\lim_{h\downarrow 0} x_{t-h}$ in E_m to[2] $x_t = \lim_{h\downarrow 0} x_{t+h}$ that is a generalized Gaussian random vector with mean vector $F_m^{m'} \lim_{h\downarrow 0} x_{t-h} + g_m^{m'}$ and a positive semi-definite covariance matrix $V_m^{m'}$, where $F_m^{m'}$, $g_m^{m'}$, and $V_m^{m'}$ are a vector and matrices with appropriate dimensions. We use the convention that $F_m^m = I$ (the identity matrix), $g_m^m = 0$ (the zero vector), and $V_m^m = 0$ (the zero matrix) for each m, thus preventing any jump within the same model.

A more precise mathematical model can be expressed as a continuous-time, time-homogeneous Markov process $(x_t, m_t)_{t\in[t_0,\infty)}$ on a hybrid state space[3] $E \stackrel{\text{def}}{=} \bigcup_{m=1}^{M} E_m \times \{m\}$ that is a formal direct-sum of Euclidean spaces E_m with generally different dimensions, with a transition probability

$$\begin{aligned} &\text{Prob}\{x_{t+h_1+h_2} \in dx', m_{t+h_1+h_2} = m' \mid x_t = x, m_t = m\} \\ &= P_{h_1+h_2}(m' \mid m) \int_{E_m} \int_{E_{m'}} \mathcal{G}(dx'; \Delta F_{m'}(h_2)x'' \\ &\quad + \Delta g_{m'}(h_2), \Delta V_{m'}(h_2)) \mathcal{G}(dx''; F_m^{m'} x''' + g_m^{m'}, V_m^{m'}) \\ &\quad \mathcal{G}(dx'''; \Delta F_m(h_1)x + \Delta g_m(h_1), \Delta V_m(h_1)) + o(h_1 + h_2) \end{aligned} \tag{2}$$

for each $(m, m') \in \{1,\ldots,M\}^2$, each $x \in E_m$, each $t \in [t_0, \infty)$, and $h_1, h_2 > 0$, where,[4] for each m and $h \geq 0$, $\Delta F_m(h) \stackrel{\text{def}}{=} e^{A_m h}$, $\Delta g_m(h) \stackrel{\text{def}}{=} \int_0^h e^{A_m \tau} b_m d\tau$, and $\Delta V_m(h) \stackrel{\text{def}}{=} \int_0^h e^{A_m \tau} Q_m e^{A_m^T \tau} d\tau$ with $Q_m = B_m B_m^T$. $\mathcal{G}(\cdot; \bar{\xi}, V)$ is the symbol for the generic generalized Gaussian distribution with mean vector $\bar{\xi}$ and a positive semi-definite covariance matrix V, of compatible dimensions, defined by its

[1] This conference paper received *Fusion 2015 Jean-Pierre Le Cadre Best Paper Award.*

[2] We assume the right-continuity to eliminate any ambiguity.

[3] Since $E = \mathbb{R}^n \times \{1,\ldots,M\}$ if $E_m = \mathbb{R}^n$ for all $m \in \{1,\ldots,M\}$, our choice of the state space provides a proper extension to the usual models used for multiple-model formulations, with $\mathbb{R} = (-\infty, \infty)$.

[4] By X^T we mean the transpose of a vector or a matrix X.

characteristic function as

$$\int e^{\sqrt{-1}\zeta^T\xi}\mathcal{G}(d\xi;\bar{\xi},V) = \exp\left(\sqrt{-1}\bar{\xi}^T\zeta - \tfrac{1}{2}\zeta^T V\zeta\right) \quad (3)$$

for each vector ζ with the dimension determined by the parameter pair $(\bar{\xi},V)$.

The discrete time observations, $y_1, y_2, y_3, \ldots$, are modeled as

$$y_k = H_{m_{t_k}k}x_{t_k} + \eta_k \quad (4)$$

for each $k = 1,2,3,\ldots$, with the time sequence, $t_1, t_2, t_3, \ldots$, such that $t_0 \le t_k < t_{k+1}$ for each k, with observation matrices, $(H_{mk})_{m=1}^M$, $k = 1,2,3,\ldots$, of appropriate dimensions,[5] and with zero-mean independent Gaussian vectors $\eta_1, \eta_2, \eta_3, \ldots$, with covariance matrices[6] $R_k = \mathbb{E}(\eta_k\eta_k^T)$. The independent initial condition at the initial time t_0 is given as,

$$\text{Prob}\{x_{t_0} \in dx, m_{t_0} = m\} = p_{m0}\mathcal{G}(dx;\bar{x}_{m0},\bar{V}_{m0}) \quad (5)$$

with an initial model probability p_{m0}, mean vector $\bar{x}_{m0}$, and positive definite covariance matrix $\bar{V}_{m0}$, for each model $m \in 1,\ldots,M$.

Then the filtering problem defined by eqns. (1) to (5) is the problem of characterizing the a posteriori probability distribution, expressed by $\hat{p}_{mk} = \text{Prob}\{m_{t_k} = m \mid y_1,\ldots,y_k\}$ and $\text{Prob}\{x_{t_k} \in dx_{t_k} \mid m_{t_k} = m, y_1,\ldots,y_k\}$ for each $m \in \{1,\ldots,M\}$, and $k = 1,2,3,\ldots$. It would be extremely difficult (if not impossible) to express $\text{Prob}\{x_{t_k} \in dx_{t_k} \mid m_{t_k} = m, y_1,\ldots,y_k\}$ in any analytical (closed) form because of the infinitly many possibilities of how the system jumps occur, in any given interval $[t_{k-1}, t_k]$. However, as shown in the next section, the continuous-time evolution of the model probability $\hat{p}_{mk}$, and the first and the second moments of the posterior state probability distribution, $\text{Prob}\{x_{t_k} \in dx_{t_k} \mid m_{t_k} = m, y_1,\ldots,y_k\}$, given model m, i.e., $\hat{x}_{mk} = \mathbb{E}(x_{t_k} \mid m_{t_k} = m, y_1,\ldots,y_k)$ and $\hat{V}_{mk} = \mathbb{E}(x_{t_k}x_{t_k}^T \mid m_{t_k} = m, y_1,\ldots,y_k) - \hat{x}_{mk}\hat{x}_{mk}^T$, can be analytically derived from eqns. (1) to (5), by a single vector homogeneous linear differential equation, as shown in the next section, Section 3.

Instead of modeling the continuous model switching by a stochastic differential equation driven by a Poisson process and a Wiener process, as formulated in [2, 8, 24], we have introduced a continuous-time Markov process on a hybrid space $\bigcup_{m=1}^M E_m \times \{m\}$, rather than $E = R^n \times \{1,\ldots,M\}$, explicitly by a transition probability defined by (1) and (2), thereby extending the general continuous-time IMM models described in [7, 24, 25]. Moreover, we explicitly model any jump between the state spaces E_m and $E_{m'}$ with generally different dimensions, by a general affine jump, $x'_{m'} = F_m^{m'}x_m + g_m^{m'} + (V_m^{m'})^{1/2}\xi_m^{m'}$, from model m to m', with zero-mean unit-variance Gaussian random vector $\xi_m^{m'}$ to represent uncertainty in the jump. By doing so, we avoid the bias issues addressed in [26], which arise when state spaces with different dimensions are handled by adding artificial zero state components and applying the standard IMM mixing algorithm mechanically.

[5] Such that $H_{mk} \in R^{d_k \times \dim(E_m)}$ for every $m \in \{1,\ldots,M\}$ where d_k is the dimension of y_k, for every k.

[6] $\mathbb{E}$ is the symbol for the conditional and unconditional mathematical expectation operators.

3. IMM ALGORITHM

First we consider the extrapolation step, generally following [14]. To do so, we define a semi-group of linear functionals $\mathcal{T}_h$ on the space $\mathcal{C}$ of all the real-valued bounded continuous functions ϕ on the hybrid space E by, $\mathcal{T}_h\phi(x,m) = \mathbb{E}(\phi(x_{t+h}, m_{t+h}) \mid x_t = x, m_t = m)$ for each $(x,m) \in E$, $t \in [t_0,\infty)$ and $h \ge 0$. Since (x_t, m_t) is a time-homogeneous Markov process, the definition does not depend on t. Then the *infinitesimal generator* $\mathcal{A}$ of $\mathcal{T}_h$ can be defined as

$$\begin{aligned}\mathcal{A}\phi(x,m) &= \lim_{h\downarrow 0} h^{-1}(\mathcal{T}_h\phi(x,m) - \phi(x,m)) \\ &= \frac{\partial}{\partial x}\phi(x,m)(A_m x + b_m) + \frac{1}{2}\text{trace}\left(\frac{\partial^2}{\partial x^2}\phi(x,m)Q_m\right) \\ &\quad + \sum_{m'=1}^M c_{mm'}\int_{E_{m'}}\phi(x',m')\mathcal{G}(dx';F_m^{m'}x + g_m^{m'}, V_m^{m'})\end{aligned} \quad (6)$$

More precisely, when the limit $\lim_{h\downarrow 0} h^{-1}(\mathcal{T}_h\phi - \phi)$ exists in the sup-norm of $\mathcal{C}$, we say the functional ϕ belongs to the domain of $\mathcal{A}$, i.e., $\phi \in \text{Dom}(\mathcal{A})$, and the last expression of eqn. (6) is uniquely implied[7] by eqns. (1) and (2). Then, for any $\phi \in \text{Dom}(\mathcal{A})$, we have [11]

$$\begin{aligned}&\mathbb{E}(\phi(x_{t+h}, m_{t+h}) \mid (x_t, m_t)) \\ &\quad = \phi(x_t, m_t) + \mathbb{E}\left(\int_t^{t+h}\mathcal{A}\phi(x_t, m_t)d\tau \mid (x_t, m_t)\right)\end{aligned} \quad (7)$$

With the (unconditional) expectation of both sides of (7), under a regularity condition that allows us to interchange the state-space expectation and the time-integral, we have

$$\mathbb{E}(\phi(x_{t+h}, m_{t+h})) = \mathbb{E}(\phi(x_t, m_t)) + \int_t^{t+h} E(\mathcal{A}\phi(x_t, m_t))d\tau \quad (8)$$

or $(d/dt)\mathbb{E}(\phi(x_t, m_t)) = \mathbb{E}(\mathcal{A}\phi(x_t, m_t))$.

Let us define $\bar{p}_{mk}(t) = \text{Prob}\{m_t = m \mid y_1,\ldots,y_k\}$, $\bar{x}_{mk}(t) = \mathbb{E}(x_t \mid m_t = m, y_1,\ldots,y_k)\bar{p}_{mk}(t)$, and $\bar{S}_{mk}(t) = \mathbb{E}(x_t x_t^T \mid m_t = m, y_1,\ldots,y_k)\bar{p}_{mk}(t)$, for each $m \in \{1,\ldots,M\}$. Then it follows from (1), (2), and (8), that, for each $t \in [t_k, t_{k+1}]$, with C defined as the $M \times M$ matrix whose (i,j) element is c_{ij} defined in (1),

$$[\bar{p}_{1k}(t)\ldots\bar{p}_{Mk}(t)] = [\hat{p}_{1k}\ldots\hat{p}_{Mk}]\exp(C(t - t_k)) \quad (9)$$

[7] See Appendix A for the derivation of (6) from (1) and (2).

$$\frac{d}{dt}\bar{x}_{mk}(t) = A_m\bar{x}_{mk}(t) + b_m\bar{p}_{mk}(t) + \sum_{m'=1}^{M} c_{m'm}(F_{m'}^m\bar{x}_{m'k}(t) + g_{m'}^m\bar{p}_{m'k}(t)) \tag{10}$$

and

$$\frac{d}{dt}\bar{S}_{mk}(t) = A_m\bar{S}_{mk}(t) + \bar{S}_{mk}(t)A_m^T + b_m\bar{x}_{mk}(t)^T + \bar{x}_{mk}(t)b_m^T + Q_m\bar{p}_{mk}(t) + \sum_{m'=1}^{M} c_{m'm}(F_{m'}^m\bar{S}_{m'k}(t)(F_{m'}^m)^T + F_{m'}^m\bar{x}_{m'k}(t)(g_{m'}^m)^T + g_{m'}^m\bar{x}_{m'k}(t)^T(F_{m'}^m)^T + (g_{m'}^m(g_{m'}^m)^T + V_m^{m'})\bar{p}_{m'k}(t)) \tag{11}$$

The initial conditions for (10) and (11) are given as $\bar{x}_{mk}(t_k) = \hat{x}_{mk}\hat{p}_{mk}$ and $\bar{S}_{mk}(t_k) = (\hat{V}_{mk} + \hat{x}_{mk}\hat{x}_{mk}^T)\hat{p}_{mk}$. Eqn. (9) is a well-known formula, while the derivation of eqns. (10) and (11) are given in Appendix B.

For each $t \in [t_k, t_{k+1}]$, let $\Xi_t = (\bar{p}_{mk}(t), \bar{x}_{mk}(t), \bar{S}_{mk}(t))_{m=1}^M$ and let φ be the function that arranges all the elements in Ξ_t into a vector in the N-dimensional Euclidean space,[8] with $N = \sum_{m=1}^{M}(1 + \dim(E_m) + \dim(E_m) \cdot (\dim(E_m) + 1)/2)$. Then, since all the equations (9) to (11) are linear ordinary differential equations, we have

$$\varphi(\Xi_t) = \exp(D(t - t'))\varphi(\Xi_{t'}) \tag{12}$$

for any (t, t') such that $t_k \le t' \le t \le t_{k+1}$, where D is an $N \times N$ matrix uniquely defined by eqns. (9) to (11), and can be calculated by any one of the known effective numerical methods.

Furthermore, if we assume $\bar{p}_{mk}(t) > 0$ for any $m \in \{1,\ldots,M\}$ and $t \in [t_k, t_{k+1}]$, it follows from (9) to (11) that

$$\frac{d}{dt}\tilde{V}_{mk}(t) = A_m\tilde{V}_{mk}(t) + \tilde{V}_{mk}(t)A_m^T + Q_m\bar{p}_{mk}(t) + \sum_{m'=1}^{M} c_{m'm}(F_{m'}^m\tilde{V}_{m'}(t)(F_{m'}^m)^T + \bar{p}_{m'k}(t)(V_{m'}^m + \Delta_{m'}^m(t)\Delta_{m'}^m(t)^T)) \tag{13}$$

with

$$\tilde{V}_{mk}(t) \stackrel{\text{def}}{=} \mathbb{E}\left(\left(x_t - \frac{\bar{x}_{mk}(t)}{\bar{p}_{mk}(t)}\right)\left(x_t - \frac{\bar{x}_{mk}(t)}{\bar{p}_{mk}(t)}\right)^T \middle| m_t = m, y_1, \ldots, y_k\right)\bar{p}_{mk}(t) \tag{14}$$

[8]We only need the values for the upper triangle elements for each symmetric matrix $\bar{S}_{km}(t)$.

and

$$\Delta_{m'}^m(t) \stackrel{\text{def}}{=} \bar{p}_{mk}(t)^{-1}\bar{x}_{mk}(t) - \bar{p}_{m'k}(t)^{-1}F_{m'}^m\bar{x}_{m'k}(t) - g_{m'}^m = \bar{x}_{mk}(t) - (F_{m'}^m\bar{x}_{m'k}(t) + g_{m'}^m) \tag{15}$$

We should note that, in (13) to (15), we have $V_m^m = 0$ and $\Delta_m^m = 0$, for each m.

The IMM update step, which precedes each extrapolation step described above, is performed by the standard IMM update formula. Namely, for each $m \in \{1,\ldots,M\}$, assuming $\bar{p}_{m(k-1)}(t_k) > 0$, we have

$$\hat{x}_{mk} = \frac{\bar{x}_{m(k-1)}(t_k)}{\bar{p}_{m(k-1)}(t_k)} + K_{mk}\left(y_k - H_{mk}\frac{\bar{x}_{m(k-1)}(t_k)}{\bar{p}_{m(k-1)}(t_k)}\right) \tag{16}$$

$$\hat{V}_{mk} = (I - K_{mk}H_{mk})\bar{V}_{mk} \tag{17}$$

where

$$\bar{V}_{mk} = \frac{\bar{S}_{m(k-1)}(t_k)}{\bar{p}_{m(k-1)}(t_k)} - \left(\frac{\bar{x}_{m(k-1)}(t_k)}{\bar{p}_{m(k-1)}(t_k)}\right)\left(\frac{\bar{x}_{m(k-1)}(t_k)}{\bar{p}_{m(k-1)}(t_k)}\right)^T \tag{18}$$

and

$$K_{mk} = \bar{V}_{mk}H_{mk}^T\mathcal{S}_{mk}^{-1} \tag{19}$$

with

$$\mathcal{S}_{mk} = H_{mk}\bar{V}_{mk}H_{mk}^T + R_k \tag{20}$$

$$\hat{p}_{mk} = \left(\sum_{m'=1}^{M} L_{m'k}\right)^{-1} L_{mk} \tag{21}$$

and

$$L_{mk} = \frac{\bar{p}_{m(k-1)}(t_k)}{\sqrt{\det(2\pi\mathcal{S}_{mk})}}\exp\left(-\frac{1}{2}\left\|y_k - \frac{\bar{x}_{m(k-1)}(t_k)}{\bar{p}_{m(k-1)}(t_k)}\right\|_{\mathcal{S}_{mk}^{-1}}^2\right) \tag{22}$$

The matrix H_{mk} in eqns. (16)–(20) is the observation matrix and R_k is the covariance matrix of the observation noise η_k, both used to define the measurement equation (4).

A critical step to develop a very simple solution in the form of the linear ordinary differential eqn. (12) is our use of the particular form of the first and the second moments, $\bar{x}_{mk}(t)$ and $\bar{S}_{mk}(t)$, rather than a usual choice of conditional mean and covariance, $\mathbb{E}(x_t \mid m_t, y_1, \ldots, y_k)$ and $\mathbb{E}(x_t x_t^T \mid m_t, y_1, \ldots, y_k) - \mathbb{E}(x_t \mid m_t, y_1, \ldots, y_k)\mathbb{E}(x_t \mid m_t, y_1, \ldots, y_k)^T$. To the best of our knowledge, this fact was shown in [8] for the first time, and expanded to a general multiple-model, affine-Gaussian dynamics and jumps in [14].

4. A SIMPLE FINANCIAL MARKET MODEL

As mentioned earlier, we will model the financial market dynamics with a simple multiple-model switching system, as in [18–20]. We use three models (i.e., $M = 3$), (i) "up" ("bull"), (ii) "steady," and (iii) "down" ("bear") models. Generally, we use "u" to represent the "price" in an appropriate sense, and "v" to represent its time derivative. The three models are defined as follows:

(i) Up (Bull) Model ($m = 1$) is based on a biased Ornstein-Uhlenbeck process, defined by the affine stochastic differential equation,

$$\begin{cases} du_t = v_t dt \\ dv_t = -\beta_1(v_t - \bar{v}_1)dt + \sqrt{q_1}dw_t \end{cases} \tag{23}$$

with unit-intensity Wiener process w_t, and three strictly positive parameters, $(\bar{v}_1, \beta_1, q_1)$.

(ii) Steady Model ($m = 2$) is a one-dimensional stationary stochastic process defined by

$$du_t = -\beta_0(u_t - \bar{u}_0)dt + \sqrt{q_0}dw'_t \tag{24}$$

with unit-intensity Wiener process w'_t, and three strictly positive parameters, $(\bar{u}_0, \beta_0, q_0)$.

(iii) Down (Bear) Model ($m = 3$) is another biased Ornstein-Uhlenbeck process defined by

$$\begin{cases} du_t = v_t dt \\ dv_t = -\beta_1(v_t + \bar{v}_1)dt + \sqrt{q_1}dw''_t \end{cases} \tag{25}$$

also with unit-intensity Wiener process w''_t. We can have a different set of parameters but will use the same set of parameters of Model 1 for simplicity.

Thus we have

$$A_1 = A_3 = \begin{bmatrix} 0 & 1 \\ 0 & -\beta_1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ \beta_1\bar{v}_1 \end{bmatrix} = -b_3,$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ \sqrt{q_1} \end{bmatrix}, \quad Q_1 = Q_3 = \begin{bmatrix} 0 & 0 \\ 0 & q_1 \end{bmatrix},$$

$A_2 = [-\beta_0]$, $b_2 = [\beta_0\bar{u}_0]$, $B_2 = [\sqrt{q_0}]$, and $Q_2 = [q_2]$ with $E_1 = E_3 = (-\infty, \infty)^2$ and $E_2 = (-\infty, \infty)$. With symmetry assumption, the transition probabilities of eqn. (1) are defined by

$$C = \begin{bmatrix} -c_1 & c_1 & 0 \\ c_2/2 & -c_2 & c_2/2 \\ 0 & c_1 & -c_1 \end{bmatrix} \tag{26}$$

with two parameters, $c_1 > 0$ and $c_2 > 0$. $F_1^2 = F_3^2 = [1 \ \ 0]$, $g_1^2 = g_3^2 = V_1^2 = V_1^3 = [0]$,

$$F_2^1 = F_3^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad g_2^1 = \begin{bmatrix} 0 \\ \bar{v}_1 \end{bmatrix} = -g_3^1, \quad \text{and}$$

$$V_2^1 = V_3^1 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{\sigma}_1^2 \end{bmatrix}, \quad \text{with } q_1 = 2\beta_1\sigma_1^2.$$

Then we can write eqn. (12) explicitly as

$$\frac{d}{dt}\Xi(t) = \begin{bmatrix} D_{11} & 0 & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \Xi(t) \tag{27}$$

with $\Xi(t) = [\bar{p}_{1k}(t) \ \bar{p}_{2k}(t) \ \bar{p}_{3k}(t) \ \bar{x}_k(t)^T \ \tilde{S}_k(t)^T]^T$, where $\bar{x}_k(t) = [\bar{x}_{1k}(t)^T \ \bar{x}_{2k}(t)^T \ \bar{x}_{3k}(t)^T]^T$, $\tilde{S}_k(t) = [\tilde{S}_{1k}(t)^T \ \tilde{S}_{2k}(t)^T \ \tilde{S}_{3k}(t)^T]^T$ (with the vector representations $\tilde{S}_k$ and $\tilde{S}_{mk}$ for the matrices $\bar{S}_k$ and $\bar{S}_{mk}$), $D_{11} = C^T$,

$$D_{21} = \begin{bmatrix} 0 & 0 & 0 \\ \beta_1\bar{v}_1 & c_2\bar{v}_1/2 & 0 \\ 0 & \beta_0\bar{u}_0 & 0 \\ 0 & 0 & 0 \\ 0 & -c_2\bar{v}_1/2 & -\beta_1\bar{v}_1 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} -c_1 & 1 & c_2/2 & 0 & 0 \\ 0 & -\beta_1 - c_1 & 0 & 0 & 0 \\ c_1 & 0 & -\beta_0 - c_2 & c_1 & 0 \\ 0 & 0 & c_2/2 & -c_1 & 1 \\ 0 & 0 & 0 & 0 & -\beta_1 - c_1 \end{bmatrix},$$

$$D_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1 & c_2(\bar{v}_1^2 + \bar{\sigma}_v^2)/2 & 0 \\ 0 & q_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_2(\bar{v}_1^2 + \bar{\sigma}_v^2)/2 & q_1 \end{bmatrix}, \quad D_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \beta_1\bar{v}_1 & 0 & c_2\bar{v}_1/2 & 0 & 0 \\ 0 & 2\beta_1\bar{v}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_2\bar{v}_1/2 & \beta_2\bar{v}_2 & 0 \\ 0 & 0 & 0 & 0 & 2\beta_2\bar{v}_2 \end{bmatrix}, \quad \text{and}$$

$$D_{33} = \begin{bmatrix} -c_1 & 2 & 0 & c_2/2 & 0 & 0 & 0 \\ 0 & -\beta_1 - c_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\beta_1 - c_1 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & -c_2/2 & c_1 & 0 & 0 \\ 0 & 0 & 0 & c_2/2 & -c_1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_1 - c_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\beta_1 - c_1 \end{bmatrix}$$

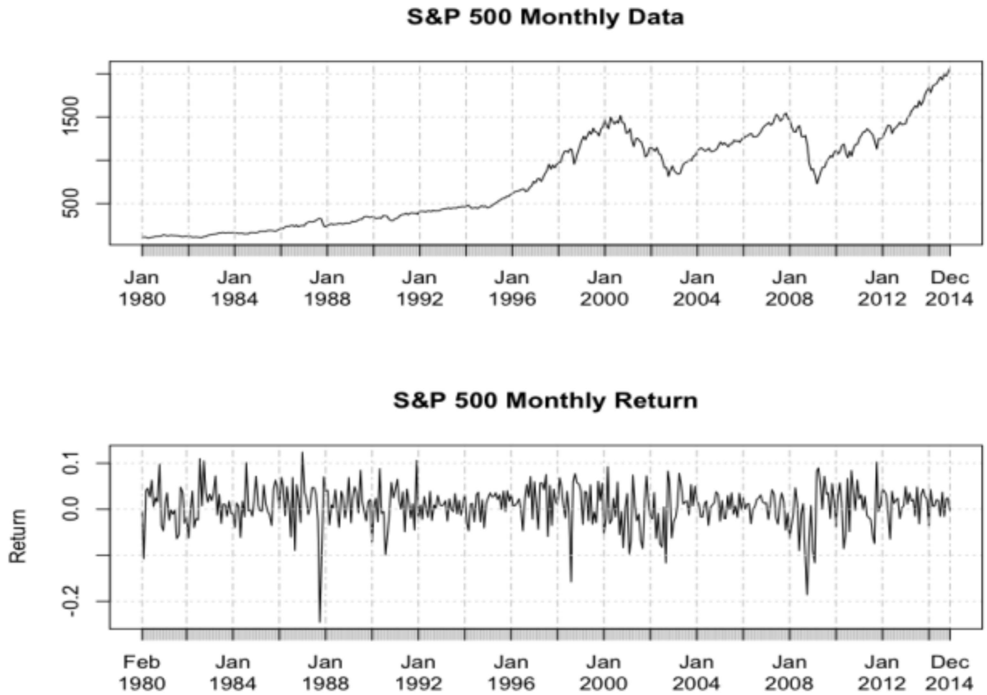

Fig. 1. Monthly S&P index form 1980 to 2014

Using the first measurement at time $t_1 = t_0$, the initial condition is given as $\bar{x}_{11}(t_1) = [y_1\ \ \bar{v}_1]^T p_{10}$, $\bar{x}_{21}(t_1) = [y_1]p_{20}$, $\bar{x}_{31}(t_1) = [y_1\ \ -\bar{v}_1]^T p_{30}$, $\bar{S}_{11}(t_1) = \mathrm{diag}(R_1, \bar{\sigma}^2_{v_1}) \cdot p_{10} + \bar{x}_{11}(t_1)\bar{x}_{11}(t_1)^T/p_{10}$, $\bar{S}_{21}(t_1) = R_1 p_{20} + \bar{x}_{21}(t_1)^2/p_{20}$, and $\bar{S}_{31}(t_1) = \mathrm{diag}(R_1, \bar{\sigma}^2_{v_1}) p_{30} + \bar{x}_{31}(t_1)\bar{x}_{31}(t_1)^T/p_{30}$, with the initial model probabilities $(p_{m0})_{m=1}^3$.

The measurement matrices are given by $H_{1k} = H_{3k} = [1\ \ 0]$ and $H_{2k} = [1]$, for all $k = 1, 2, 3, \ldots$.

5. APPLYING IMM TO FINANCIAL MODELING FOR ASSET ALLOCATION

There are two main approaches to analyze financial markets for investment and portfolio management. Fundamental analysis considers economic factors to make subjective judgments on the qualitative relationship between portfolio and market returns, whereas technical analysis uses quantitative historical data to predict future price movement. In this paper, we use the technical analysis approach where we apply the IMM model described in the previous section to model the dynamics of the equity market based on historical data.

Specifically, we focused on modeling the Standard & Poor's 500 (S&P 500) index as well as how to dynamically allocate the asset to invest in the index futures according to the model prediction. S&P 500 index is an American stock index based on the combined capitalization of 500 large companies in the US. It is one of the most widely followed benchmarks for the US and the world economy. Figure 1 shows the S&P monthly historical data from 1980 to 2014.

To test the algorithm, we randomly selected one daily, one weekly, and one monthly data sets, each with 100 data points to evaluate the performance on different time scales accordingly. In order to assess the market condition, the closing prices were used as the measurements and the three dynamic models: "up (bull)," "steady," and "down (bear)" as described in the previous section were used to model the S&P dynamics. In each test, the resulting estimated probabilities of the three models from the IMM algorithm were used to make the asset allocation decisions. The parameters were set, without any significant adjustments, as[9]: $(p_{m0})_{m=1}^3 = (0.3,\ 0.5,\ 0.2)$, $c_1 = c_2 = 1/3\ \mathrm{day}^{-1}$, $\beta_0 = \beta_1 = \beta_2 = 2\ \mathrm{day}^{-1}$, $\sigma_0 = \$20$, $\sigma_1 = \sigma_2 = 2\$/\mathrm{day}$, $\bar{u}_0 = \$1100$, and $\bar{v}_1 = -\bar{v}_2 = 4\$/\mathrm{day}$.

Traditional investment strategies usually apply heuristic rules or numerical indicators obtained from the historical data to determine the market trends. For example, stochastic oscillator (SO) or relative strength index (RSI) are well-known financial momentum indicators for contrarian investing [16]. These indicators are designed to determine the market conditions such as a potential top (resistance) or a bottom (support). A contrarian investor buys and sells against the market sentiment during a specific time based on the indicators. In that sense, one could consider the IMM algorithm developed in this paper and the resulting model probabilities as another momentum indicator. This new indicator attempts to determine the potential market overbought or oversold conditions. For example, when the "up" model probability is the highest one among the three and is above a certain threshold, it may indicate an overbought condition, and when the "down" model probability is

[9] $ represents S&P index (as a virtual unit).

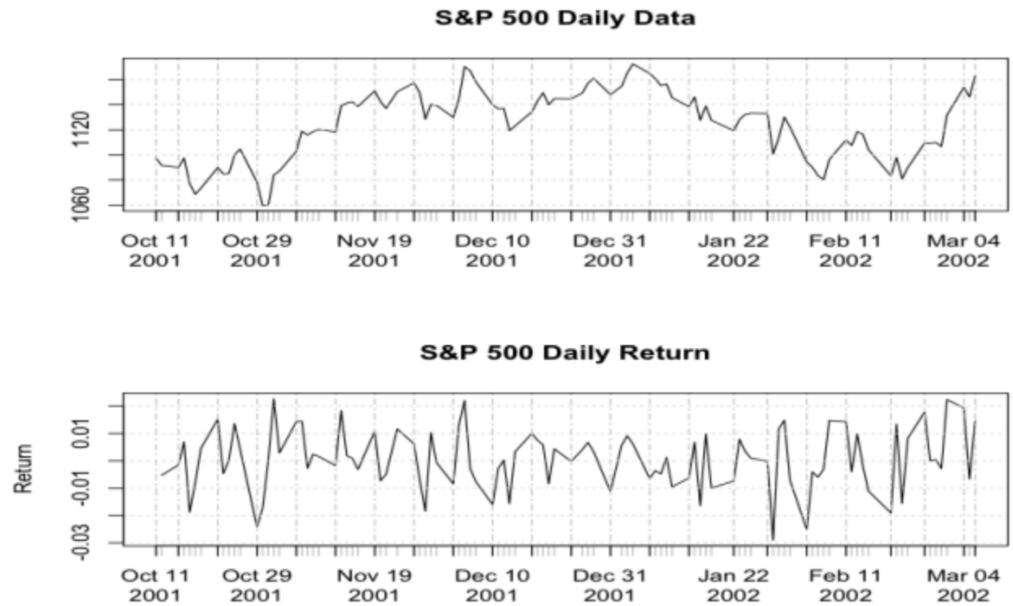

Fig. 2. S&P Daily Data—100 Days

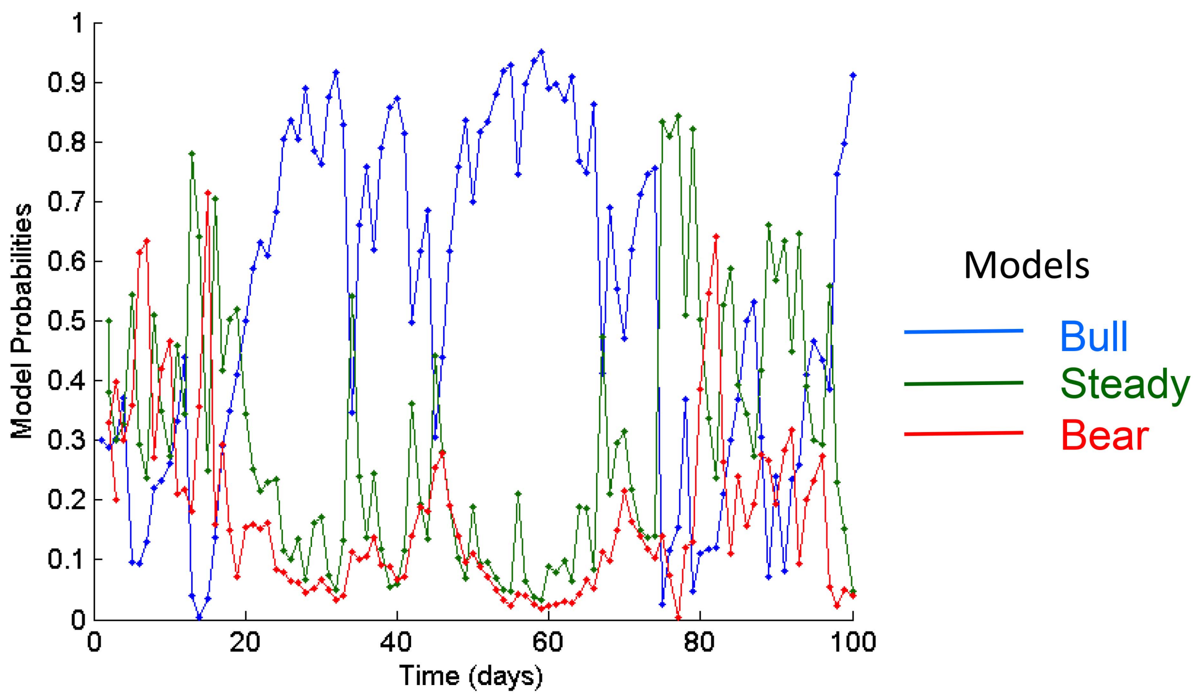

Fig. 3. IMM Model Probabilities—S&P Daily Data

the highest one and is above a certain threshold, it may indicate an oversold condition.

With the IMM indicator, we dynamically allocate the asset and make trading decisions accordingly. For example, a simple strategy is to short (sell) the S&P futures[10] when the "up" probability is the highest one (overbought) and to long (buy) when the "down" probability is the highest one. We may also want to close our positions and sit on the sideline when the market is uncertain ("steady" mode probability is the highest). However, while this "contrarian" approach could lead to higher gain than usual, it may have the opposite effect when the market is in a strong trending mode. To mitigate this risk, when the IMM "up" or "down" probabilities are in extreme values (say, > 0.95) which indicates a potential strong trend, the decision rule mentioned above will be reversed to follow the market directions.

With the above simple asset allocation rules based on the IMM indicator, we conduct simulation and test their performances on the three randomly selected S&P data

[10]S&P futures is one of the most liquid futures markets in the world. One can long or short the futures contracts as long as there is a counter party who is willing to take the opposite side.

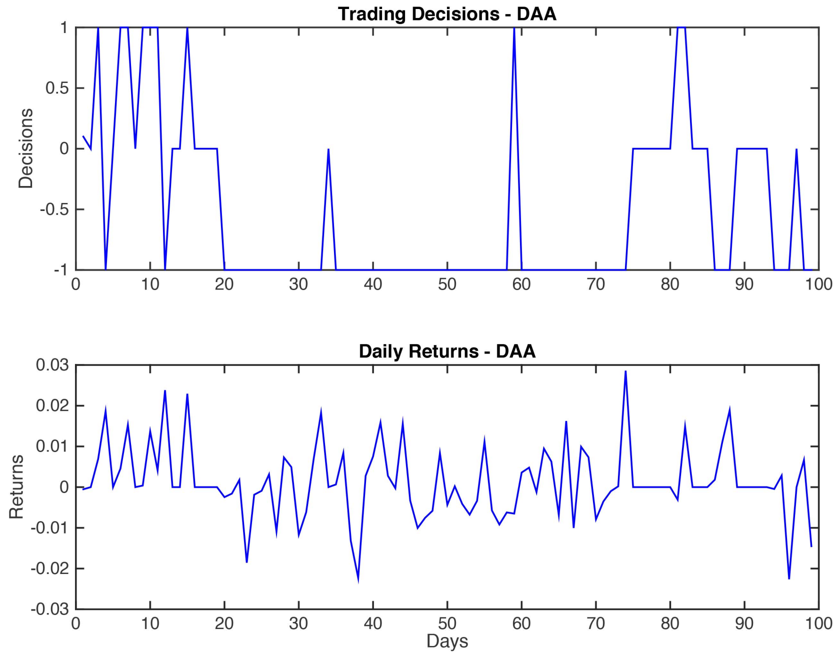

Fig. 4. IMM-DAA Trading Decisions and Daily Returns

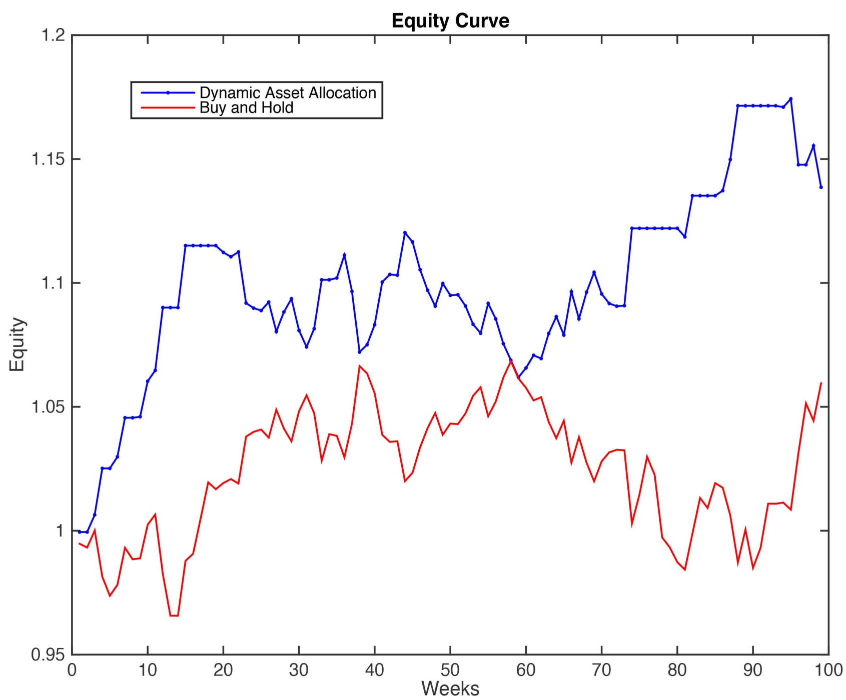

Fig. 5. Equity Curves—Buy-and-Hold vs. DAA

sets. We also compare its performance with the naïve buy-and-hold policy. Note that in the simulation, we use historical end-of-the-day S&P settlement prices to emulate the filled-prices of the transactions. We assume no transaction cost and no slippage.

1. Daily Data

Figure 2 shows a randomly selected set of daily S&P closing prices and returns over a 100-day period. The daily returns represent the daily equity percentage changes of the buy-and-hold strategy. Figure 3 shows the probability trajectories of the three models estimated by the IMM algorithm. In Figure 3, the model probabilities are shown by the blue line for the bull model ($m = 1$), the green line for the steady model ($m = 2$), and the red line for the bear model ($m = 3$).

The corresponding trading decisions of the IMM dynamic asset allocation (IMM-DAA) strategy and its daily returns are shown in Figure 4. In the figure, decision "1" represents a long position, "-1" represents

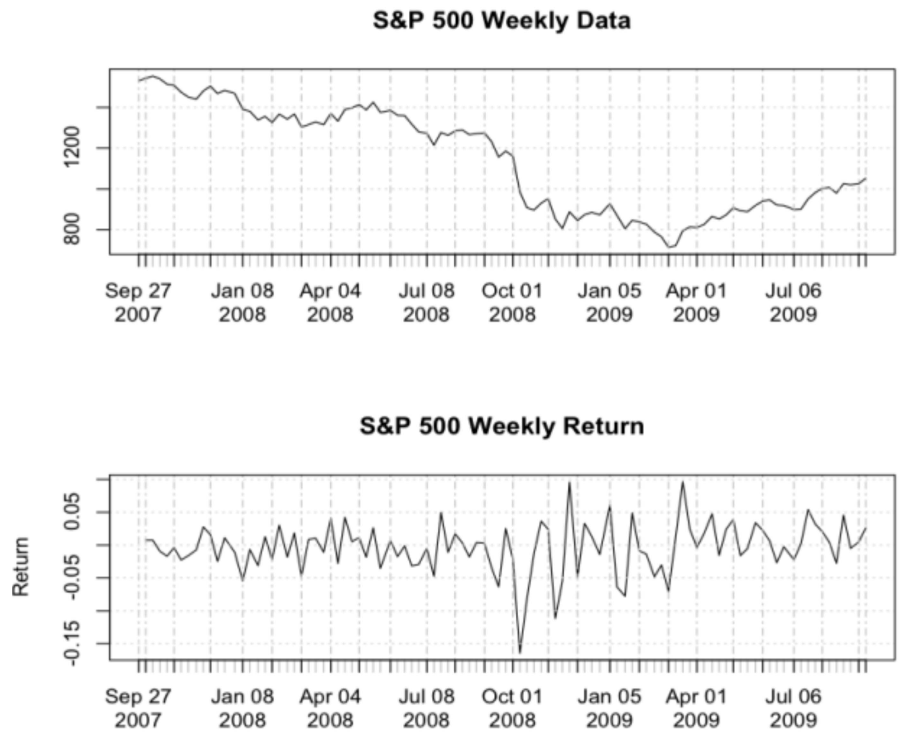

Fig. 6. S&P Weekly Data

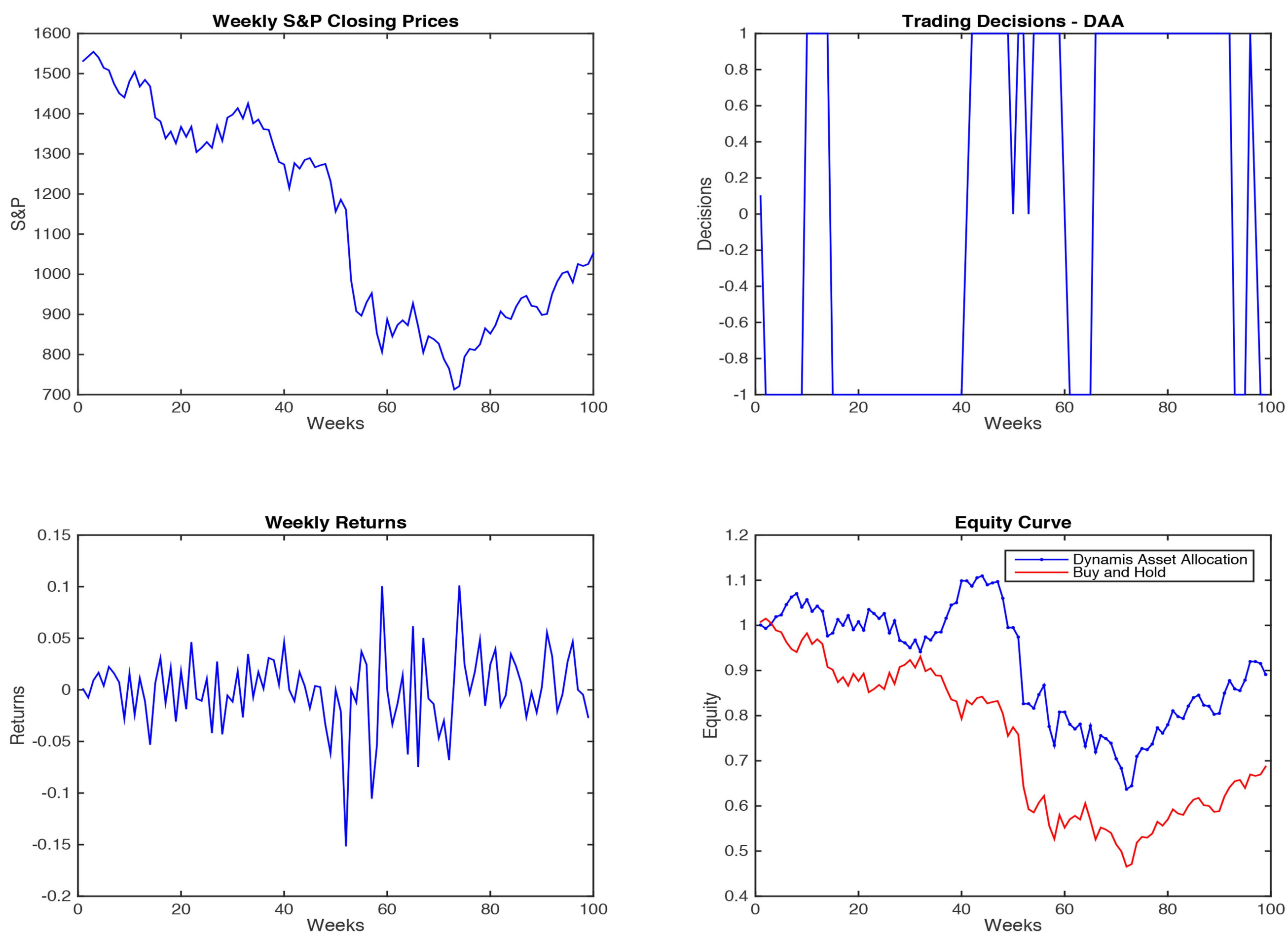

Fig. 7. Trading Performance—Weekly Data

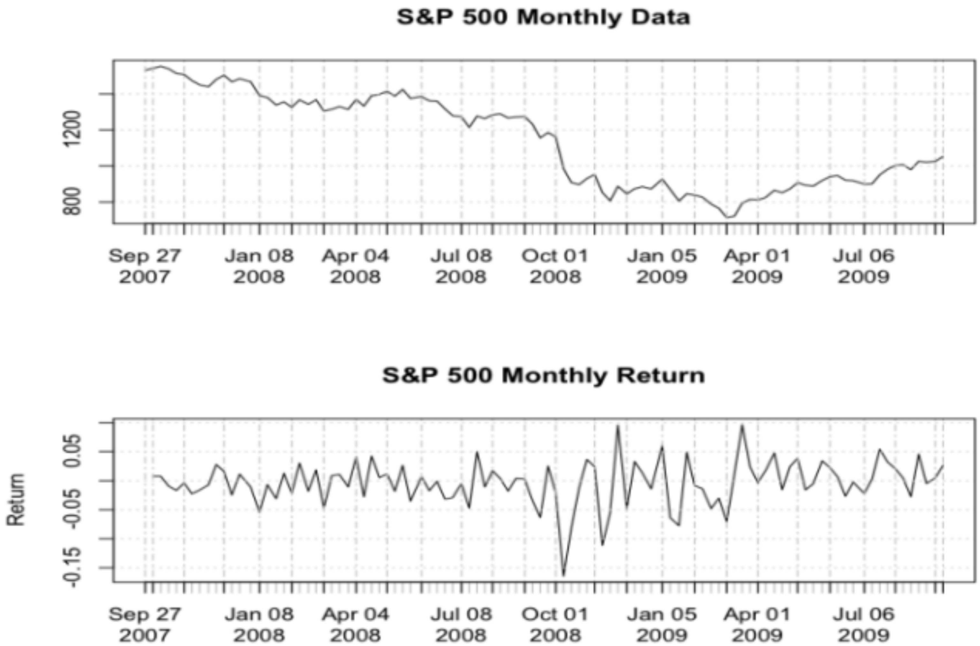

Fig. 8. S&P Monthly Data

a short position, and "0" represents no position. Figure 5 compares the equity curve over the 100-day period for the DAA strategy and the buy-and-hold (BH) policy. In the results, we assume no transaction costs or slippage.[11] As seen from the figure, DAA performs significantly better than the BH strategy with only a few trading actions—a total of around 20 over the 100-day period. At the end of the 100-day period, the cumulative return for BH is under 6% while DAA's return is almost 14%. Note that the maximum drawdown[12] of the BH policy is approximately 7% while the maximum drawdown of the DAA is only about 5%.

In Figure 3, after the jump from the lower dimensional model to a higher dimensional model, the model probability history may have some apperance of the "bias," which may be a result of a slight mismatch of the mean g_2^1 or g_2^3 with the real data.

II. Weekly and Monthly Data

Figures 6–9 show the results corresponding to the weekly and monthly data. Note that the decision rules based on the IMM indicators are exactly the same for the three data sets. Since we use the continuous-time system model, we do not have to adjust the system dynamics parameters in response to the sampling intervals, $t_{k+1} - t_k$. Based on the Markov property, the standard deviation of the process noise (volatility) is proportional to the square root of the time difference between two subsequent observations.

As shown in the figures, DAA either performs better than or close to BH with significantly lower drawdown. For example, Figure 7 shows that while BH loses about 31% of the equity over the 100 weeks period with a maximum drawdown of about 54%, DAA only loses 11% with a maximum drawdown of 43% over the same time period. Similarly, over a 100-month period, Figure 9 shows that while BH earns about 26% of the equity with a maximum drawdown of about 54%, DAA earns a slightly less return of 21% over the same period but with a significantly smaller drawdown of only 24%. Note that the randomly selected 100-month period includes the 2007–2008 credit crisis where prolonged market ups and downs exist for many months. While it is true that the rate of return and Sharpe ratio for BH are slightly better than that of DAA for this monthly time period, the maximum drawdown for BH is almost 230% higher than DAA which itself could be catastrophic. This demonstrates another potential benefit of applying the proposed DAA approach.

Table 1 summarizes the performance results for the three randomly selected data sets. In the table, an industry-standard performance indicator called the "Sharpe ratio"[13] is also presented for performance comparison. Higher Sharpe ratio indicates a better risk-adjusted return. It is clear from the table that the IMM based DAA (IMM-DAA) is an effective and promising asset allocation method.

[11] For S&P futures trading, given the liquidity and market size, the transaction cost is minimum. For example, with a standard e-mini S&P futures contract (~ $100k), the average transaction cost is less than 0.005% (< $5) of the contract size. Given that in the 100 trading periods, there were about 20 transactions, the difference is negligible (~ 0.1%).

[12] Drawdown is defined as the peak-to-trough decline during a specific period of an investment. A drawdown is usually quoted as the percentage between the peak and the trough [16].

[13] The Sharpe ratio is a measure for calculating risk-adjusted return. It is the average return earned in excess of the risk-free rate over the return volatility (standard deviation).

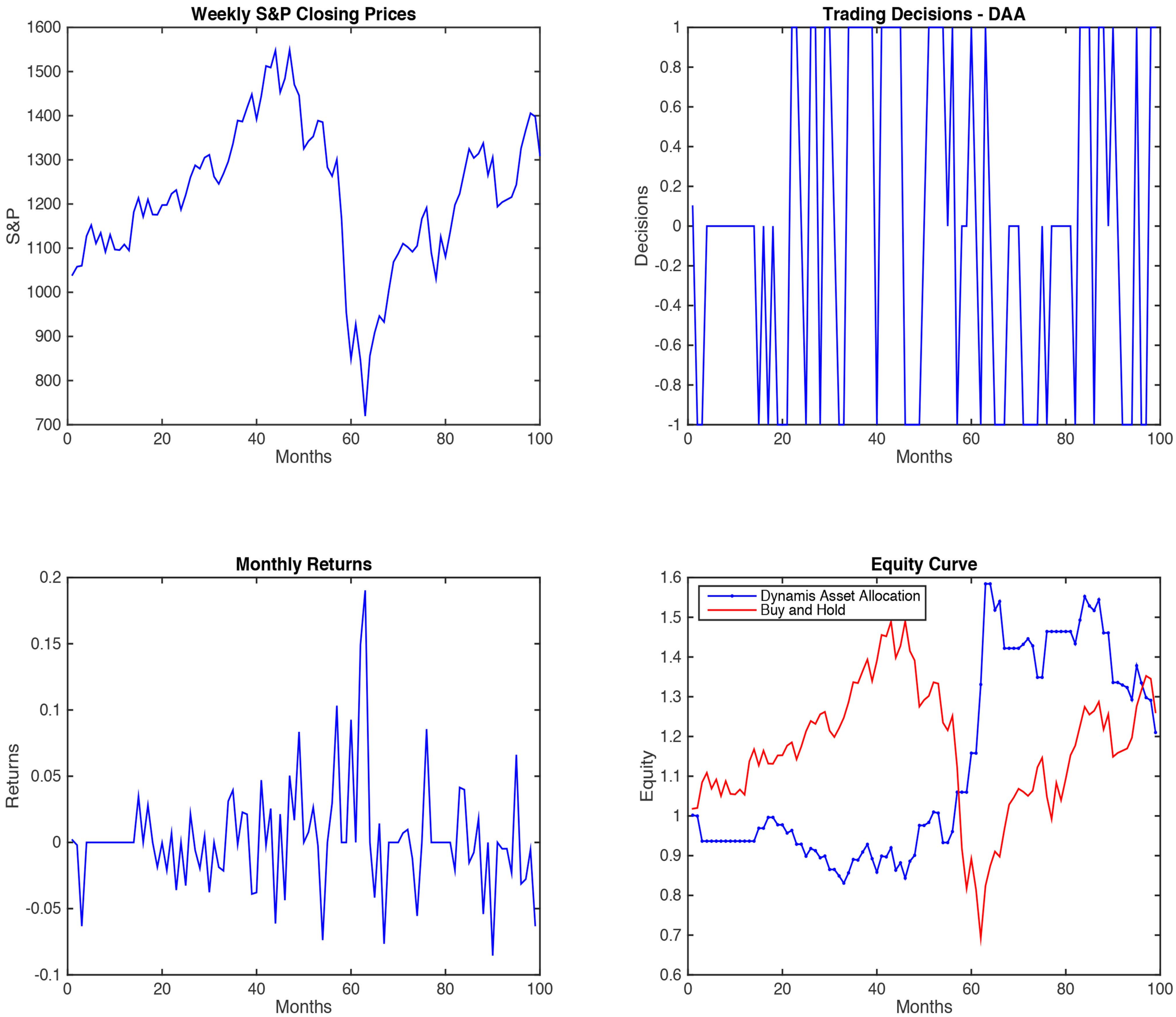

Fig. 9. Trading Performance—Monthly Data

6. CONCLUSIONS

In this paper, we presented a continuous-time, discrete-observation, Interacting Multiple Model (IMM) algorithm, based on the continuous-time IMM extrapolation developed in [14], and applied it to financial market dynamic modeling. We modeled the system by a continuous-time, jump Markov process and estimated the system state based on a sequence of discrete-time, linear-Gaussian observations. We utilized a rather naïve switching process with multiple linear stochastic system models to represent the S&P market dynamics model. The resulting IMM model probabilities serve as momentum indicators to make the dynamic asset allocation decisions (DAA). We tested the resulting IMM-DAA strategy on several randomly selected S&P data sets of various time scales. The results showed that the newly developed IMM indicator and the corresponding asset allocation strategy may have a potential to significantly outperform the baseline naïve buy-and-hold policy with lower risk.

TABLE 1
Performance Comparison

	Rate of Return	Maximum Drawdown	Sharpe Ratio
Daily—BH	5.95%	7.88%	0.878
Daily—DAA	−13.86%	5.40%	2.307
Weekly—BH	−31.26%	54.14%	−0.615
Weekly—DAA	−10.88%	42.60%	−0.134
Monthly—BH	−26.08%	53.54%	0.192
Monthly—DAA	−21.00%	23.62%	0.160

The goal of this paper is to demonstrate the potential of the new continuous-time IMM algorithm when applied to financial market analysis and asset allocation problems. To demonstrate the effectiveness of the application of the new IMM algorithm to general financial market problems requires additional testing of large amounts of real data and comparison with other methods proposed in the past, which is beyond the scope of this paper. Furthermore, in financial market modeling, consideration of non-Gaussian disturbance may be of significant interest. As seen in Sections 2 and 3, the development of our continuous-time IMM algorithm depends crucially on the Gaussian assumptions. A non-Gaussian extension of the proposed approach is therefore an interesting immediate sequel of the research of this paper.

Beyond several obvious refinements to the presented modeling approach, e.g., choosing the number of models, adjusting the intra and inter multiple model system parameters, adding "rate of model change" to the model probability itself, a potential future research direction to extend the promising preliminary work is to combine the technical approach described in this paper with a fundamental approach where both qualitative and quantitative information is utilized. Particularly, we should explore the highly relevant and emerging data fusion paradigm such as Bayesian networks and social networks for financial modeling and portfolio risk management.

APPENDIX A: CALCULATION OF INFINITESIMAL GENERATOR

For given $\phi \in \mathcal{C}$, it follows from (2) and the definition of the operator $\mathcal{T}_h$ that

$$\mathcal{T}_{h_1+h_2}\phi(x,m) = \sum_{m'=1}^{M} P_{h_1+h_2}(m' \mid m)\int_{E_{m'}} \phi(x',m') \cdot \Phi_m^{m'}(dx';x,h_1,h_2) + o(h_1+h_2) \tag{28}$$

for any $h_1,h_2 > 0$, each $m \in \{1,\ldots,M\}$ and $x \in E_m$, where $\Phi_m^{m'}(\cdot;x,h_1,h_2)$ is the convolution of the three generalized Gaussian distributions in (2). Substituting (1) into (28), we have

$$\begin{aligned}
\mathcal{T}_{h_1+h_2}&\phi(x,m) \\
&= \sum_{\substack{m'=1\\ m'\neq m}}^{M} c_{mm'}(h_1+h_2) \\
&\quad \times \int_{E_{m'}} \phi(x',m')\Phi_m^{m'}(dx';x,h_1,h_2) \\
&\quad + \left(1 - \sum_{\substack{m'=1\\ m'\neq m}}^{M} c_{mm'}(h_1+h_2)\right) \\
&\quad \times \int_{E_m} \phi(x',m)\Phi_m^{m}(dx';x,h_1,h_2) + o(h_1+h_2)
\end{aligned} \tag{29}$$

Hence, we have, for $h > 0$,

$$\begin{aligned}
&h^{-1}(\mathcal{T}_h\phi(x,m) - \phi(x,m)) \\
&\quad = \sum_{m'=1}^{M} c_{mm'}\int_{E_{m'}} \phi(x',m')\mathcal{G}_{m'}(dx';F_m^{m'}x + g_m^{m'}, V_m^{m'}) \\
&\quad + h^{-1}\left(\int_{E_m} \phi(x',m)\mathcal{G}_m(dx';e^{A_m h}x + b_m, \Delta V_m(h)) - \phi(x,m)\right) \\
&\quad + o(h)
\end{aligned} \tag{30}$$

It is well known (e.g., cf. [11]) that the second term of the right hand side converges to $(\partial/\partial x)\phi(x,m)(A_m x + b_m) + \frac{1}{2}\mathrm{trace}((\partial^2/\partial x^2)\phi(x,m)Q_m)$, and (5) follows.

For a fixed pair (h_1,h_2), eqn. (2) implies the model transition from m to m' happens at most one time in the time interval $[t, t+h_1+h_2]$ at time $t+h_1$. Usual IMM practice (e.g., cf. [4] or [6]) is to let $h_1 = 0$ and use a time interval $h = h_2$ that is equal to the sensor revisit time, and to use a Gaussian approximation.[14] In [14], a multiple-model extrapolation algorithm where two or more model transitions are possible within a given extrapolation time interval was developed analytically without sub-dividing the extrapolation interval, which inevitably involves Gaussian approximation for each subinterval. Instead, the extrapolation algorithm developed in [14] and described in Appendix B preserves exact moment calculations by (8) and (10). At the end of the extrapolation interval, however, we need a Gaussian approximation to apply the IMM updating step, as seen in Section 3.

APPENDIX B: MOMENT CALCULATIONS

For a fixed $m \in \{1,\ldots,M\}$ and a fixed $i \in \{1,\ldots,\dim(E_m)\}$, define ϕ by $\phi(x,m') = x_i$ if $m' = m$, 0 otherwise. Then substituting this ϕ into eqn. (5), we have

$$\mathcal{A}\phi(x_t,m_t) = \delta_{m_t m}(A_m x_t + b_m)_i + c_{m_t m}(F_{m_t}^m x_t + g_{m_t}^m)_i \tag{31}$$

for each t. Taking expectation of (31) leads to

$$\begin{aligned}
&\mathbb{E}(\mathcal{A}\phi(x_t,m_t)) \\
&\quad = \sum_{m'=1}^{M} \mathbb{E}(\mathcal{A}\phi(x_t,m_t) \mid m_t = m')\mathrm{Prob}\{m_t = m'\} \\
&\quad = \sum_{m'=1}^{M} p_t(m')\mathbb{E}(\delta_{m'm}(A_m x_t + b_m)_i \\
&\qquad + c_{m'm}(F_{m'}^m x_t + g_{m'}^m)_i \mid m_t = m') \\
&\quad = p_t(m)(A_m\bar{x}(t\mid m) + b_m)_i + \sum_{m'=1}^{M} p_t(m')c_{m'm}(F_{m'}^m\bar{x}(t\mid m') + g_{m'}^m)_i \\
&\quad = (A_m\bar{x}(t;m) + b_m p_t(m))_i + \sum_{m'=1}^{M} c_{m'm}(F_{m'}^m\bar{x}(t;m') + g_{m'}^m p_t(m'))_i
\end{aligned} \tag{32}$$

[14] In the IMM literature, this Gaussian approximation is often referred to as *mixing*, which is also characterized as *interacting* among multiple models.

from which eqn. (10) follows, with $p_t(m') = \mathrm{Prob}\{m_t = m'\}$, $x(t \mid m') = \mathbb{E}(x_t \mid m_t = m')$ and $x(t;m') = x(t \mid m') \times p_t(m')$, for every $m' \in \{1,\ldots,M\}$.

The ordinary differential equation (11) for the non-centric second moments, $S(t;m) = S(t \mid m)p_t(m)$ with $S(t \mid m) = \mathbb{E}(x_t x_t^T \mid m_t = m)$, can be obtained in a similar way, using ϕ defined as, for a fixed $m \in \{1,\ldots,M\}$ and a pair (i,j) such that $(i,j) \in \{1,\ldots,\dim(E_m)\}^2$, $\phi(x,m') = x_i x_j$ if $m' = m$, zero otherwise.

In order to obtain eqn. (13), we should first note $S(t;m) = V(t;m) + \bar{x}(t;m)\bar{x}(t;m)^T p_t(m)^{-1}$, which implies

$$\dot{S}(t;m) = \dot{V}(t;m) - \bar{x}(t;m)\bar{x}(t;m)^T p_t(m)^{-2}\dot{p}_t(m) + (\dot{\bar{x}}(t;m)\bar{x}(t;m)^T + \bar{x}(t;m)\dot{\bar{x}}(t;m)^T)p_t(m)^{-1} \tag{33}$$

with $\dot{S}(t;m) = (d/dt)S(t;m)$, $\dot{V}(t;m) = (d/dt)V(t;m)$, $\dot{\bar{x}}(t;m) = (d/dt)\bar{x}(t;m)$, and $\dot{p}_t(m) = (d/dt)p_t(m) = \sum_{m'=1}^{M} p_t(m')c_{m'm}$. Then, eqn. (13) is obtained by substituting eqns. (10), (11) and (15) into (33).

We should note that in order to derive the first and the second moments through eqns. (10) and (11), to be precise, we need one highly technical step, because, for example, $\phi(x,m) = x_i$ if $m = m'$, 0 otherwise, does not define a bounded functional ϕ on $\bigcup_{m=1}^{M} E_m \times \{m\}$. To justify the use of eqns. (6) to (8), we may need to consider a series of stopped processes, each bounded by a compact set $\{(x,m) \mid \|x\| \le k\}$ for each integer k, and to apply Dynkin's lemma to obtain the desired result as a limit, as is done in [12] and [13].

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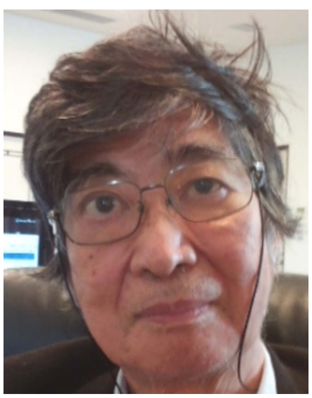

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