Filtering Solution to the Out-of-Sequence Measurement Problem with Colored and Correlated Noise

Antje Westenberger
Research and Advanced Engineering Team Sensor Fusion
Daimler AG, Ulm, Germany
antje.westenberger@daimler.com

Marc Muntzinger
Research and Advanced Engineering Team Active Sensors
Daimler AG, Ulm, Germany
marc.muntzinger@daimler.com

Klaus Dietmayer
Institute of Measurement, Control, and Microtechnology
Ulm University, Germany
klaus.dietmayer@uni-ulm.de

Abstract—In multi-sensor fusion, information from different sensors with asynchronous cycle durations and different latencies is fused. This often leads to an out-of-sequence problem, i.e., the original order of the measurements is lost. The optimal filtering solution to these out-of-sequence problems has been derived under the assumption of white and uncorrelated process and measurement noise [2]. However, in realistic scenarios, these assumptions are often not fulfilled. Therefore a new filter for the colored noise problem is derived. In addition, a new filtering solution to the out-of-sequence problem is proposed that can handle colored as well as correlated process and measurement noise.

I. INTRODUCTION

Modern sensor fusion systems use information from an increasing amount of different sensors. On the one hand, there are sensors that take a trigger or can be synchronized. But on the other hand, many sensors are free-running and have sensor-specific cycle durations. In addition, there is often a sensor-specific delay between the timestamp when a measurement was taken and the arrival timestamp at the fusion system, the so-called latency. This latency depends on different factors, such as the preprocessing and the communication transfer. Since these delays may vary between different sensors, measurements from one sensor often arrive at the fusion unit with a longer delay than measurements from another sensor. This loss of temporal order is a so-called out-of-sequence problem, which can be observed in most sensor fusion systems.

Figure 1 shows an example of an out-of-sequence problem. Here two sensors with two different latencies are depicted. Since the latency of sensor one is longer than the latency of sensor two, the first out-of-sequence measurement (OOSM) is the measurement that was taken at timestep $t_{k_0}$. It is older than the measurement taken at timestep $t_k$, but arrives later at the fusion unit.

These out-of-sequence measurements pose a problem in usual filtering algorithms such as the Kalman filter because the original order of the measurements is usually needed. The optimal filtering solution to the out-of-sequence problem, the so-called retrodiction, has been proposed by Bar-Shalom [2]. However, this filtering solution assumes white and uncorrelated process and measurement noise. Although this assumption is made very often, it is not fulfilled in many practical applications.

First, there are a lot of examples where the process noise is colored. In automotive applications, a constant velocity model is widely-used. The acceleration is therefore modelled in the process noise. But since acceleration is auto-correlated, a constant velocity model in automotive contexts always has to deal with colored process noise.

Further, colored measurement noise is observed in many applications. For example, the measurement frequency of modern radars is often higher than the noise bandwidth, therefore the measurement noise cannot be assumed white [17]. As soon as several sensors with a similar measurement principle are used, e.g. two radar sensors, their measurement noise is correlated as well.

Finally, the process and measurement noise can be correlated. An example is the measuring of wind speed on an airplane. If winds are buffeting the plane, the gusts of wind influence both the process (the airplane dynamics) and the measurement (the sensed wind speed) [12].

Due to all these practical applications where process and measurement noise are colored or correlated, a method to deal with out-of-sequence measurements in these cases is needed. Therefore a new filter for the colored measure-
ment noise is derived in the following sections. Based on this colored noise filter, new algorithms for out-of-sequence measurements treatment are derived that do not depend on the assumptions of white and uncorrelated process and measurement noise.

The outline of the paper is as follows: Section II gives an overview of related publications. In Section III, the problem is stated and described mathematically. The main derivations of the new algorithms for out-of-sequence measurements with colored process noise, colored measurement noise and correlated process and measurement noise are given in Sections IV, V and VI, respectively. Results from a Monte Carlo simulation are given in Section VII. Section VIII discusses strategies and determination of the sensor-specific latencies can be found in [10], [11]. Details on synchronization methods and determination of the sensor-specific latencies can be found, e.g., in [7], [16], [6].

The problem of optimal filtering with colored noise can be solved first by Bar-Shalom [3]. A second solution based on a forward-prediction approach was published in [9]. Since then, many publications have dealt with out-of-sequence problems in different contexts. An application of out-of-sequence algorithms in an automotive pre-crash system can be found in [10], [11]. Details on synchronization strategies and determination of the sensor-specific latencies can be found, e.g., in [7], [16], [6].

The problem of optimal filtering with colored noise can be solved by augmenting the state vector in the Kalman filter [5]. Since this state augmentation works well for colored process noise, but yields numerical problems with colored measurement noise, a different approach called measurement differencing was proposed by Bryson [4]. This approach models the colored noise as an autoregressive process. Methods to adaptively estimate the corresponding autoregressive parameters were given in [17] and [1]. A similar technique for the continuous-time filter with autocorrelated measurement noise was given in [13].

The measurement differencing approach by Bryson is widely-used in practice, but has several disadvantages. It uses two consecutive measurements to define a pseudo measurement that eliminates the colored noise. This yields additional temporal delays, since a single measurement cannot be incorporated in the state estimation. These additional delays must be avoided in time-critical contexts. Further, since two consecutive measurements are needed, the handling of single out-of-sequence measurements poses a complex problem that has not been solved yet. Another approach that does not formulate an autoregressive process, but directly estimates the cross covariance between state and colored noise was published by Wendel [14], [15]. This approach does not work well in practice (as will be shown in Section VII) because the dynamics of the colored measurement noise are not considered in the derivation except for the cross covariance. Therefore a new filter for the colored measurement noise case is derived in the following sections that is based on the underlying idea from Wendel, but estimates the colored noise in addition to the state.

III. PROBLEM FORMULATION

A general time-varying dynamic system can be modeled via the state equation

\[ x_k = F_{k,k-1} x_{k-1} + v_{k,k-1}. \]  

(1)

This describes the dynamics from timestep \( t_{k-1} \) to timestep \( t_k \), where \( x_k \) is the state vector at time \( t_k \), \( F_{k,k-1} \) is the system matrix and \( v_{k,k-1} \) is the process noise that is often assumed gaussian distributed and white,

\[ v_{k,k-1} \sim N(0, Q_{k,k-1}), \]

\[ E[v_{k,k-1}v_{j,j}^T] = Q_{k,k-1}\delta_{k,j}, \]  

(2)

with the standard Kronecker delta \( \delta_{k,j} \). The measurement equation is given as

\[ z_k = H_k x_k + w_k, \]  

(3)

where \( z_k \) denotes a measurement at time \( t_k \), \( H_k \) is the measurement matrix from the state space to the measurement space and \( w_k \) is the measurement noise that is usually assumed gaussian distributed, white and independent from the process noise \( v_{k,k-1} \).

\[ w_k \sim N(0, R_k), \]

\[ E[w_kw_j^T] = R_k\delta_{k,j}, \]

\[ E[v_{k,k-1}w_j^T] = 0. \]  

(4)

The optimal filtering solution to this formal problem was given by Kalman [8]. The Kalman filter assumes the correct temporal order of all the measurements. However, as mentioned before, the chronological order of the measurements can not be guaranteed. Let \( z_{k|k} \) denote the Kalman estimation at timestep \( t_k \) after the update with \( z_k \). An out-of-sequence problem occurs if after the update with \( z_k \), an older measurement \( z_{k_0} \) taken at a timestep \( t_{k_0} < t_k \) arrives at the fusion unit. This out-of-sequence measurement cannot be incorporated in the state estimation using standard Kalman filtering methods.

The optimal solution to the out-of-sequence problem was deduced by Bar-Shalom [2]. The algorithm consists of a retrodiction, a kind of backward-prediction to the out-of-sequence timestamp, and an OOSM innovation. However, these algorithms assume white and uncorrelated process and measurement noise. Therefore new generalized retrodiction algorithms for colored and correlated noise are derived in the following sections.
IV. COLORED PROCESS NOISE

In many contexts the process noise is not white, but temporally correlated. The formal problem is then

\[
\begin{align*}
x_k &= F_{k-1}x_{k-1} + v_{k-1}, \\
z_k &= H_k x_k + w_k, \\
v_{k-1} &= \Phi_{k-1}v_{k-1} + \varepsilon_{k-1}, \\
\varepsilon_k &\sim N(0, Q_{\varepsilon_k}), \\
w_k &\sim N(0, R_k), \\
E[v_{k-1}v_{k-1}^T] &= \Phi_{k-1}Q_{k-1}, \\
E[w_kw_k^T] &= R_k \delta_{k,j}, \\
E[\varepsilon_k\varepsilon_j^T] &= Q_{\varepsilon_k} \delta_{k,j}, \\
E[w_k\varepsilon_j^T] &= 0.
\end{align*}
\]

The process noise is therefore itself output of a linear system.

The colored process noise problem can simply be solved using an augmented state approach [5]. The augmented system equation is

\[
\begin{bmatrix} x_k \\ v_{k-1} \end{bmatrix} = \begin{bmatrix} F_k & I \\ 0 & \Phi_{k-1} \end{bmatrix} \begin{bmatrix} x_{k-1} \\ v_{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_{k-1} \end{bmatrix}.
\]

\( \Leftrightarrow \quad x'_k = F'_k x'_{k-1} + v'_{k-1}. \)

This augmented system has white process noise \( v'_{k-1} \), therefore the standard retrodiction following Bar-Shalom [2] can be applied directly.

V. COLORED MEASUREMENT NOISE

In practical applications, the measurement noise \( w_k \) is often not white, but colored noise. In this case the system and measurement equations are

\[
\begin{align*}
x_k &= F_{k-1}x_{k-1} + v_{k-1}, \\
z_k &= H_k x_k + w_k, \\
v_{k-1} &= \Phi_{k-1}v_{k-1} + \varepsilon_{k-1}, \\
\varepsilon_k &\sim N(0, Q_{\varepsilon_k}), \\
w_k &\sim N(0, R_k), \\
E[v_{k-1}v_{k-1}^T] &= \Phi_{k-1}Q_{k-1}, \\
E[w_kw_k^T] &= R_k \delta_{k,j}, \\
E[\varepsilon_k\varepsilon_j^T] &= Q_{\varepsilon_k} \delta_{k,j}, \\
E[w_k\varepsilon_j^T] &= 0.
\end{align*}
\]

The colored cross covariance problem often yields numerical problems, as the measurement noise covariance gets singular. Therefore a different approach is used in this paper: Instead of state augmentation, the influence of the colored noise is estimated explicitly by estimating the cross covariance between state and measurement noise in addition to the usual state and covariance estimation. This approach was proposed by Wendel [15]. The derivation is based on the idea from Wendel [15], but directly estimates the colored measurement noise and corresponding measurement error covariance matrix as well. Therefore let \( \hat{w}_{k|k-1} \) and \( \hat{w}_k \) denote the a priori and a posteriori measurement noise estimations, respectively, with corresponding covariance matrices \( R_{k|k-1} \) and \( R_k \). The prediction of the process noise is straightforward following the system dynamics (7):

\[
\hat{w}_{k|k-1} = \Psi_{k-1}\hat{w}_{k|k-1|k-1} + Q_{\varepsilon_k}. \quad (8)
\]

with the corresponding estimation error covariance

\[
R_{k|k-1} = \Psi_{k-1}R_{k|k-1|k-1}\Psi_{k-1} + Q_{\varepsilon_k}. \quad (9)
\]

Note that if the measurement noise is autocorrelated, the measurement noise is also correlated with the state estimation [15]. Thus the cross covariance of state and measurement noise has to be estimated, which is defined as

\[
P_{k|k-1} = E\left[ (x_k - \hat{x}_{k|k-1})(w_k - \hat{w}_{k|k-1})^T \right]. \quad (10)
\]

This cross covariance can be predicted to the next timestep as follows:

\[
P_{k|k-1} = E\left[ (F_{k-1}(x_k - \hat{x}_{k|k-1|k-1}) + v_{k-1}) \cdot (\Psi_{k-1}(w_k - \hat{w}_{k|k-1}) + \varepsilon_{k-1})^T \right] = F_{k-1}P_{k-1|k-1}^x\Psi_{k-1}. \quad (11)
\]

For the innovation, the Kalman gain

\[
K_k = P_{k|k-1}^xS_k^{-1}, \quad (12)
\]

with colored measurement noise has to be derived, which looks slightly different from the white noise Kalman filter. Here \( P_{k|k-1}^x \) denotes the cross covariance between state and measurement, and \( S_k \) is the covariance of the residual

\[
\gamma_k = z_k - \hat{z}_{k|k-1} = z_k - H_kx_{k|k-1} - \hat{w}_{k|k-1}. \quad (13)
\]

The cross covariance between state and measurement noise reads

\[
P_{k|k-1}^x = E\left[ (x_k - \hat{x}_{k|k-1})(z_k - \hat{z}_{k|k-1})^T \right] = E\left[ (x_k - \hat{x}_{k|k-1})(H_k(x_k - \hat{x}_{k|k-1}) \right. \\
\left. + (w_k - \hat{w}_{k|k-1})^T \right] = P_{k|k-1}H_k^T + P_{k|k-1}^w. \quad (14)
\]
The residual covariance is
\[ S_k := E \left( (z_k - \hat{z}_{k|k-1}) (z_k - \hat{z}_{k|k-1})^T \right) \]
\[ = E \left[ H_k \left( x_k - \hat{x}_{k|k-1} \right) + (w_k - \hat{w}_{k|k-1}) \right] \]
\[ = H_k P_{k|k-1} H_k^T + R_{k|k-1} + H_k P_{w} w H_k^T \]
\[ = H_k P_{k|k-1} H_k^T + R_{k|k-1} + H_k P_{w} w H_k^T, \] (15)

The innovation of the estimation error covariance matrix then follows from the fundamental equations of linear estimation (see [3], p. 128):
\[ P_k = P_{k|k-1} - K_k \left( P_{w} w H_k \right)^T. \] (16)

The ansatz for the innovation of the estimated measurement noise follows from the fundamental equations as well:
\[ \hat{w}_k = \hat{w}_{k|k-1} + \text{cov} (w_k, z_k) Z^{k-1} \text{cov} (z_k) (Z^{k-1})^{-1} \gamma_k. \]

The cross covariance between measurement noise and measurement can be deduced as
\[ \text{cov} (w_k, z_k) Z^{k-1} = E \left[ (w_k - \hat{w}_{k|k-1}) \left( H_k (x_k - \hat{x}_{k|k-1}) \right) \right] \]
\[ = R_{k|k-1} + (P_{w} w H_k)^T H_k^T. \] (17)

Therefore the innovation of the estimated measurement noise is
\[ \hat{w}_k = \hat{w}_{k|k-1} + \left( R_{k|k-1} + (P_{w} w H_k)^T H_k^T \right) S_k^{-1} \gamma_k. \]

The innovation of the corresponding estimation error covariance can be calculated as follows:
\[ R_{k|k} = R_{k|k-1} - \left( R_{k|k-1} + (P_{w} w H_k)^T H_k^T \right) S_k^{-1} \]
\[ \cdot \left( R_{k|k-1} + H_k P_{w} w H_k^T \right). \] (18)

Similar to the additional cross covariance prediction (11), the innovation of \( P_{w} w H_k \) has to be derived:
\[ P_{w} w H_k := E \left[ (x_k - \hat{x}_{k|k}) (w_k - \hat{w}_{k|k}) \right] \]
\[ = (I - K_k H_k) P_{k|k-1} w H_k + K_k R_{k|k-1}. \] (19)

This completes the filter derivation for the colored measurement noise case.

B. Summary of the Colored Measurement Noise Kalman Filter

The filtering equations derived in the previous section for the case of colored measurement noise can be summarized as follows:

Prediction:
\[ \hat{x}_{k|k} = F_k x_{k-1} \hat{x}_{k-1|k-1}, \]
\[ \hat{w}_k = \Psi_k \hat{w}_{k-1|k-1}, \]
\[ P_{k|k} = F_k \left( P_{k|k-1} - K_k H_k \right) F_k^T + Q_k, \]
\[ R_{k|k} = \Psi_k \left( R_{k|k-1} - K_k H_k \right) \Psi_k^T + Q_k \]
\[ P_{w} w H_k := F_k \left( P_{w} w H_k \right) F_k^T + Q_k. \] (20)

C. State Retrodiction

Now a new retrodiction algorithm for the case of autocorrelated measurement noise is derived. Let \( \hat{x}_{k|k} \) be the state estimation after innovation with all measurements up to \( z_k \),
\[ \hat{x}_{k|k} = E \left[ x_k | z_1, \ldots, z_k \right]. \] (21)

Suppose there arrives an out-of-sequence measurement \( z_{k_0} \) from an earlier timestamp \( t_{k_0} \). The retrodiction first estimates the state at time \( t_{k_0} \) incorporating all the measurements up to \( z_k \), but without the OOSM \( z_{k_0} \):
\[ \hat{x}_{k_0|k} = E \left[ x_{k_0} | z_1, \ldots, z_k \right]. \] (22)

From the system equation (7), the ansatz for the retrodiction can be deduced as follows like in the standard retrodiction [2]:
\[ \hat{x}_{k_0|k} = F_{k_0|k}^{-1} \left( \hat{x}_{k|k} - \hat{v}_{k, k_0} \right). \] (23)

Note that the a posteriori process noise estimation \( \hat{v}_{k, k_0} \) is not zero because the innovation with \( z_k \) has already been done. Therefore the process noise has to be estimated in
addition to the state. From the fundamental equation of linear estimation ([3], p. 128), the following equation holds:

\[ \hat{v}_{k,\xi_{k} | k} = \text{cov} \left( v_{k,\xi_{k} | Z} | Z_{k} \right) \]

\[ = \text{cov} \left( v_{k,\xi_{k} | Z} \right) \text{cov} \left( z_{k} | Z_{k} \right)^{-1} \gamma_{k}, \quad (24) \]

where \( Z_{k} \) denotes the set of all pseudo measurements up to and including measurement \( z_{k} \):

\[ Z_{k} := \{ z_{1}, ..., z_{k} \}. \quad (25) \]

The following equation holds:

\[ \text{cov} \left( v_{k,\xi_{k} | Z_{k} | Z_{k-1}} \right) \]

\[ = \text{cov} \left( v_{k,\xi_{k} | Z} \right) + \text{cov} \left( v_{k,\xi_{k} | Z} \right) \text{cov} \left( v_{k,\xi_{k} | Z} \right)^{-1} \gamma_{k}. \quad (26) \]

since all other terms are orthogonal. Substitution of (26) and (15) into Equation (24) yields the complete state retrodiction with colored measurement noise:

\[ \hat{x}_{k_{0} | k} = F_{k_{0} | k}^{-1} \left( \hat{x}_{k} - Q_{k_{0} | k} H_{k}^{-1} H_{k}^{-1} + R_{k} \right) + H_{k} P_{k_{0} | k}^{-1} + \left( P_{k_{0} | k}^{-1} \right)^{-1} \gamma_{k}. \quad (27) \]

D. Covariance Retrodiction

Like in the standard retrodiction [2], the retrodicted covariance has the form:

\[ P_{k_{0} | k} = \text{cov} \left( x_{k} - v_{k,\xi_{k} | Z} | Z_{k} \right) \]

\[ = F_{k_{0} | k}^{-1} \left[ \text{cov} \left( x_{k} | Z_{k} \right) - \text{cov} \left( x_{k}, v_{k,\xi_{k} | Z} | Z_{k} \right) \right] \]

\[ - \text{cov} \left( x_{k}, v_{k,\xi_{k} | Z} | Z_{k} \right)^{T} + \text{cov} \left( v_{k,\xi_{k} | Z} | Z_{k} \right)^{T} \]

\[ = \left( F_{k_{0} | k}^{-1} \right)^{-1} P_{k_{0} | k} - P_{k_{0} | k} X_{k_{0} | k} - \left( P_{k_{0} | k} X_{k_{0} | k} \right)^{T} \]

\[ + P_{k_{0} | k}^{T} \left( F_{k_{0} | k}^{-1} \right)^{T}. \quad (28) \]

Therefore the additional covariance matrices \( P_{k_{0} | k} P_{x_{k_{0} | k}} \) and \( P_{k_{0} | k}^{T} \) have to be deduced. From the fundamental equation of linear estimation [3],

\[ \text{cov} \left( \left( \begin{array}{c} x_{k} \cr v_{k,\xi_{k} | k} \end{array} \right) | Z_{k} \right) = \text{cov} \left( \left( \begin{array}{c} x_{k} \cr v_{k,\xi_{k} | k} \end{array} \right) | Z_{k}^{T} \right) \]

\[ - \text{cov} \left( \left( \begin{array}{c} x_{k} \cr v_{k,\xi_{k} | k} \end{array} \right) | Z_{k} \right) \text{cov} \left( z_{k} | Z_{k} \right)^{-1} \gamma_{k} \]

\[ \text{cov} \left( z_{k} | Z_{k} \right) \gamma_{k} \]. \quad (29) \]

The cross covariance between state vector and process noise can be derived as follows:

\[ \text{cov} \left( x_{k}, v_{k,\xi_{k} | Z} | Z_{k} \right) = \text{cov} \left( \left( F_{k,\xi_{k} | k} \left( x_{k} - \hat{x}_{k_{0} | k} \right) + v_{k,\xi_{k} | k} \right) | Z_{k} \right) \]

\[ = Q_{k_{0} | k}. \quad (30) \]

The cross covariance between state and measurement in the case of colored measurement noise was given in Equation (14). Substituting Equations (30), (14) and (26) into Equation (29) finally yields the covariances \( P_{k_{0} | k} \) and \( P_{k_{0} | k}^{T} \) as follows:

\[ \text{cov} \left( \left( \begin{array}{c} x_{k} \cr v_{k,\xi_{k} | k} \end{array} \right) | Z_{k} \right) \]

\[ = \left[ \begin{array}{c|c} P_{k_{0} | k} & Q_{k_{0} | k} \end{array} \right] - \left[ \begin{array}{c|c} P_{k_{0} | k} H_{k}^{-1} + P_{k_{0} | k} H_{k}^{-1} \end{array} \right] S_{k}^{-1} \]

\[ = \left[ \begin{array}{c|c} P_{k_{0} | k} & P_{x_{k_{0} | k}} \end{array} \right] - \left[ \begin{array}{c|c} P_{k_{0} | k} H_{k}^{-1} + P_{k_{0} | k} H_{k}^{-1} \end{array} \right] S_{k}^{-1} \]

\[ = \left[ \begin{array}{c} P_{k_{0} | k} P_{x_{k_{0} | k}} \end{array} \right] \quad (31) \]

The sought-after covariances are therefore

\[ P_{k_{0} | k} = Q_{k_{0} | k} \left( P_{k_{0} | k} H_{k}^{-1} + P_{k_{0} | k} H_{k}^{-1} \right) S_{k}^{-1} \]

\[ + \left( P_{k_{0} | k} H_{k}^{-1} + P_{k_{0} | k} H_{k}^{-1} \right) S_{k}^{-1} \]

\[ P_{k_{0} | k} = Q_{k_{0} | k} - Q_{k_{0} | k} H_{k}^{-1} S_{k}^{-1} H_{k} Q_{k_{0} | k}. \quad (32) \]

Substitution of these covariance matrices into Equation (28) finally yields the retrodiction of the estimation error covariance with colored measurement noise.

E. Measurement Noise Retrodiction

The estimated measurement noise \( \hat{w}_{k_{0} | k} \) with the covariance \( R_{k_{0} | k} \) has to be retrodicted as well. The derivation is similar to the retrodiction of \( \hat{x}_{k_{0} | k} \) and \( P_{k_{0} | k} \) and is therefore omitted here. The results are

\[ \hat{w}_{k_{0} | k} = \Psi_{k_{0} | k}^{-1} \left( \hat{w}_{k_{0} | k} \right) Q_{k_{0} | k} S_{k}^{-1} \gamma_{k}, \]

\[ R_{k_{0} | k} = \Psi_{k_{0} | k}^{-1} \left( R_{k_{0} | k} \right) Q_{k_{0} | k} \Psi_{k_{0} | k}^{-1} \quad (33) \]

with the corresponding cross covariances

\[ P_{k_{0} | k} = Q_{k_{0} | k} \left( P_{k_{0} | k} H_{k}^{-1} + R_{k_{0} | k} \right) S_{k}^{-1} \]

\[ P_{k_{0} | k} = Q_{k_{0} | k} - Q_{k_{0} | k} S_{k}^{-1} Q_{k_{0} | k}. \quad (34) \]

F. Cross Covariance Retrodiction

In addition to the state and estimation error covariance retrodiction, when the measurement noise is colored, the cross covariance between state and measurement noise has to be retrodicted as well, similar to the filter equations (20). The retrodiction of the cross covariance can be deduced as follows:

\[ P_{k_{0} | k} = \text{cov} \left( \hat{x}_{k_{0} | k}, w_{k_{0}} \right) \]

\[ = \text{cov} \left( \left( F_{k_{0} | k} \left( x_{k} - \hat{x}_{k_{0} | k} \right) \right) \Psi_{k_{0} | k}^{-1} \left( w_{k} - \varepsilon_{k_{0}} \right) \right) \]

\[ = F_{k_{0} | k}^{-1} P_{x_{k_{0} | k}} \left( \Psi_{k_{0} | k}^{-1} \right)^{T}. \quad (35) \]

This completes the retrodiction of state, measurement noise and corresponding covariances with autocorrelated measurement noise.
The OOSM Innovation

In the following, the OOSM innovation of state, covariance and cross covariance is derived. The ansatz for the update equation is the same as in the standard retrodiction [2]:

\[
\hat{x}_{k|k} = \hat{x}_{k|k} + W_{k|0} (z_{k|0} - H_{k|0} \hat{x}_{k|0}) - \hat{w}_{k|0},
\]

(36)

with a special OOSM gain matrix

\[
W_{k|0} := \text{cov} (\hat{x}_{k|k}, z_{k|0}) Z^k \text{cov} (z_{k|0} | Z^k)^{-1}.
\]

(37)

Here the cross covariance between the state and the out-of-sequence measurement is

\[
\text{cov} (x_{k|k}, z_{k|0} | Z^k) = \text{cov} \left( x_{k|k}, H_{k|0} F_{k|k|0}^{-1} (x_{k} - v_{k|0}) + w_{k|0} | Z^k \right)
\]

(38)

\[
\begin{align*}
&= \left( P_{k|k} - P_{k|k|0} \right) F_{k|k|0}^{-1} H_{k|0}^T + \text{cov} (x_{k}, \Psi_{k|0} w_{k - \varepsilon_{k|0}} | Z^k) \\
&= \left( P_{k|k} - P_{k|k|0} \right) F_{k|k|0}^{-1} H_{k|0}^T + \text{cov} (x_{k}, \Psi_{k|0} w_{k - \varepsilon_{k|0}} | Z^k)
\end{align*}
\]

and therefore

\[
W_{k|0} = P_{k|k|0}^{-1} S_{k|0}^{-1}.
\]

The OOSM innovation of the corresponding covariance follows from the fundamental equation of linear estimation [3]

\[
P_{k|k} = P_{k|k} - P_{k|k|0} S_{k|0}^{-1} \left( P_{k|k|0} \right)^T.
\]

(40)

The OOSM-innovation of the measurement noise and the corresponding covariance is similar, therefore the derivation is omitted here. The result is

\[
\begin{align*}
\hat{w}_{k|0} &= \hat{w}_{k|0} + P_{k|k|0} S_{k|0}^{-1} \gamma_{k|0}, \\
R_{k|0} &= R_{k|0} - P_{k|k|0} S_{k|0}^{-1} P_{k|k|0}^T
\end{align*}
\]

with the cross variance

\[
P_{k|k|0} = \Psi_{k|0} \left( P_{k|k|0} \right)^T H_{k|0}^T + \Psi_{k|0} R_{k|0}.
\]

(42)

Finally the OOSM-innovation of the cross covariance between state and measurement noise can be derived as follows:

\[
P_{k|k|0} = E \left[ (x_{k} - \hat{x}_{k|0}) (w_{k} - \hat{w}_{k|0})^T \right]
\]

\[
= E \left[ (x_{k} - \hat{x}_{k|k}) - \hat{w}_{k|0} (H_{k|0} (x_{k} - \hat{x}_{k|k})
\]

\[
+ (w_{k|0} - \hat{w}_{k|0}) | \Psi_{k|0} (w_{k|0} - \hat{w}_{k|0}) + \varepsilon_{k|0}) \right]
\]

\[
= P_{k|k|0} - W_{k|0} H_{k|0} P_{k|k|0} \Psi_{k|0} - W_{k|0} R_{k|0} \Psi_{k|0}.
\]

(43)

This completes the OOSM retrodiction algorithm for colored measurement noise, which is summarized in the next section.

H. Summary of the Colored Measurement Noise Retrodiction

The derived algorithm for the out-of-sequence measurements treatment in the case of autocorrelated measurement noise can be summarized as follows:

\[
\begin{align*}
\hat{x}_{k|0} &= F_{k|0}^{-1} \left[ \hat{x}_{k|k} - Q_{k|k} H_{k|0}^T S_{k|0}^{-1} \gamma_{k|0} \right], \\
\hat{w}_{k|0} &= \Psi_{k|0} \left( \hat{w}_{k|0} - Q_{k|k} S_{k|0}^{-1} \gamma_{k|0} \right), \\
P_{k|k|0} &= Q_{k|k} - P_{k|k-1} H_{k|0}^T + P_{k|k-1} S_{k|0}^{-1} H_{k|0} Q_{k|k}, \\
P_{k|k|0} &= Q_{k|k} - Q_{k|k} H_{k|0}^T S_{k|0}^{-1} H_{k|0} Q_{k|k}, \\
P_{k|k|0} &= P_{k|k} - P_{k|k|0} - P_{k|k|0} \Psi_{k|0} P_{k|k|0}^T \\
\Psi_{k|0} &= P_{k|k|0} - P_{k|k|0} \Psi_{k|0} P_{k|k|0}^T.
\end{align*}
\]

VI. Correlated Process and Measurement Noise

Another violation of the assumptions in the usual retrodiction algorithm [2] is correlated process and measurement noise at the same timestep. In this case the formal problem reads

\[
\begin{align*}
x_{k} &= F_{k,k-1} x_{k-1} + v_{k,k-1}, \\
z_{k} &= H_{k} x_{k} + w_{k}, \\
v_{k,k-1} &\sim N (0, Q_{k,k-1}), \\
w_{k} &\sim N (0, R_{k}), \\
E \left[ v_{k,k-1} v_{k,j}^T \right] &= Q_{k,k-1} \delta_{k,j}, \\
E \left[ w_{k,w_{k}}^T \right] &= R_{k} \delta_{k,j}, \\
E \left[ v_{k,k-1} w_{k,j}^T \right] &= M_{k,k-1} \delta_{k,j}.
\end{align*}
\]

(44)
The filtering solution to this correlated noise problem is derived in [12] and is restated here. The prediction equations are as usual:

\[
\hat{x}_{k|k-1} = F_{k,k-1} \hat{x}_{k-1|k-1}, \\
P_{k|k-1} = F_{k,k-1} P_{k-1|k-1} F_{k,k-1}^T + Q_{k,k-1}.
\] (45)

The innovation is slightly different from the standard Kalman filter:

\[
K_k = (P_{k|k-1} H_k^T + M_{k,k-1}) (H_k P_{k|k-1} H_k^T + H_k M_{k,k-1} + M_{k,k-1} H_k^T + R_k)^{-1},
\]

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - H_k \hat{x}_{k|k-1}),
\]

\[
P_{k|k} = P_{k|k-1} - K_k (H_k P_{k|k-1} + M_{k,k-1}^T). \tag{46}
\]

In the following the retrodiction for this modified filter algorithm is derived.

**A. State Retrodiction**

For the state retrodiction with correlated process and measurement noise, the same ansatz as in Section V-C is used:

\[
\hat{x}_{k|0} = F_{k,k|0}^{-1} (\hat{x}_{k|0} - \hat{v}_{k,k|0}) = F_{k,k|0}^{-1} (\hat{x}_{k|0} - \text{cov}(\nu_{k,k|0}, z_k) Z_k^{k-1} S_k^{k-1} \gamma_k). \tag{47}
\]

The cross covariance between the process noise and the measurement now differs from the standard case without correlated noise. It can be deduced as follows:

\[
\text{cov}(\nu_{k,k|0}, z_k) = \text{cov}(\nu_{k,k|0}, H_k (F_{k,k|0} x_{k|0} + \nu_{k,k|0}) + w_k Z_k^{k-1}) = \text{cov}(\nu_{k,k|0}, Z_k^{k-1} H_k^T) + \text{cov}(\nu_{k,k|0}, w_k Z_k^{k-1}) = Q_{k,k|0} H_k^T + M_{k,k}.
\] (48)

with the covariance matrix $M_{k,k|0}$ of the correlation between $\nu_{k,k|0}$ and $w_k$ according to Equation (44). This yields the state retrodiction for systems with correlated process and measurement noise:

\[
\hat{x}_{k|0} = F_{k,k|0}^{-1} (\hat{x}_{k|0} - (Q_{k,k|0} H_k^T + M_{k,k}) S_k^{k-1} \gamma_k). \tag{49}
\]

The two components $Q_{k,k|0} H_k^T$ and $M_{k,k|0}$ reflect the influence of the process noise on the state estimation and on the measurement noise, respectively. Both influences have to be eliminated when applying the retrodiction.

**B. Covariance Retrodiction**

The retrodiction of the estimation error covariance slightly differs from the standard uncorrelated noise case [2]. With the same ansatz as (28), the covariance retrodiction can be deduced taking into account Equation (48). The derivation is similar to the colored measurement noise case and is therefore omitted here. The resulting covariance retrodiction is

\[
P_{k|0} = F_{k,k|0}^{-1} \left[ P_{k|k} - P_{k|k-1} F_{k,k|0} \right] T + P_{k|k-1} \left( F_{k,k|0} \right)^{-1} T,
\] (50)

with the covariance matrices

\[
P_{k|k|0} = Q_{k,k|0} - P_{k|k-1} H_k^T S_k^{k-1} (H_k Q_{k,k|0} + M_{k,k}) S_k^{k-1},
\]

\[
P_{k|k|0} = Q_{k,k|0} - (Q_{k,k|0} H_k^T + M_{k,k}) S_k^{k-1}, \tag{51}
\]

Note that the covariance $P_{k|k}$ in the covariance retrodiction (50) is calculated in a different way than in the standard Kalman filter according to (46).

**C. OOSM Innovation**

The OOSM innovation does not depend on the correlation between process and measurement noise, therefore the standard OOSM innovation from [2] can be applied:

\[
\hat{x}_{k|0} = \hat{x}_{k|0} + P_{k|k-1}^{zz} S_k^{k-1} (z_k - H_k \hat{x}_{k|0}),
\]

\[
P_{k|0} = P_{k|k} - P_{k|k-1}^{zz} S_k^{k-1} \left( P_{k|k-1}^{zz} \right)^T, \tag{52}
\]

with the cross covariance matrix $P_{k|k-1}^{zz}$ between state and measurement.

**D. Summary of the Retrodiction with Correlated Process and Measurement Noise**

The derived filtering solution to the correlated process and measurement noise out-of-sequence problem is summarized as follows:

**Retrodiction:**

\[
\hat{x}_{k|0} = F_{k,k|0}^{-1} \left[ \hat{x}_{k|0} - (Q_{k,k|0} H_k^T + M_{k,k}) S_k^{k-1} \gamma_k \right],
\]

\[
P_{k|0} = Q_{k,k|0} - P_{k|k-1} H_k^T S_k^{k-1} (H_k Q_{k,k|0} + M_{k,k}) S_k^{k-1},
\]

\[
P_{k|0} = Q_{k,k|0} - (Q_{k,k|0} H_k^T + M_{k,k}) S_k^{k-1}, \tag{51}
\]

Note that the covariance $P_{k|k}$ in the covariance retrodiction (50) is calculated in a different way than in the standard Kalman filter according to (46).

**OOSM innovation:**

\[
P_{k|k|0}^{zz} = \left( P_{k|k} - P_{k|k-1}^{zz} \right) \left( P_{k|k-1}^{zz} \right)^T H_k^T S_k^{k-1},
\]

\[
W_{k|0} = P_{k|k-1}^{zz} S_k^{k-1},
\]

\[
\hat{x}_{k|0} = \hat{x}_{k|0} + W_{k|0} (z_k - H_k \hat{x}_{k|0}),
\]

\[
P_{k|0} = P_{k|k} - P_{k|k-1}^{zz} S_k^{k-1} \left( P_{k|k-1}^{zz} \right)^T. \tag{52}
\]
The following section gives a comparison of different Kalman filters for the case of colored measurement noise using simulated data. Here the measurements are assumed in-sequence; the evaluation of the OOSM algorithms under correlated noise will be studied in future work. The simulation consists of a constant velocity process with white process noise, whereas the measurement noise has an autocorrelation function in the form of an exponential function. The simulation was averaged over 500 Monte Carlo runs. The performance of the newly derived filter for colored process noise is compared to several state-of-the-art filters. Figure 2 shows the results.

![Fig. 2. Position error of different filters for the colored measurement noise case using 500 Monte Carlo runs](image)

Clearly the standard Kalman filter (depicted in blue) yields inaccurate results due to the white noise assumption. The colored noise filter following Wendel [15] does not perform satisfactory as well, because the colored process noise must be estimated in addition. The newly derived filtering solution is depicted in red. This colored noise filter performs as well as the measurement differencing approach (depicted in green). However, as mentioned before, measurement differencing has the disadvantage of time delays and out-of-sequence measurements cannot be incorporated up to now.

**VIII. DISCUSSION**

The newly derived colored noise filter yields results comparable to the measurement differencing approach, but at higher computational costs, since the measurement noise with the corresponding covariances has to be estimated as well. Therefore the decision which filter to use should depend on the context. In cases where computational costs are a major drawback, but the application is less time-critical and there are no out-of-sequence measurements, the measurement differencing approach is sufficient. On the other hand, when time-criticality is a main restriction and out-of-sequence measurements occur, the proposed filtering solution including out-of-sequence measurements treatment should be preferred.

In this paper, filtering solutions to the case of colored as well as correlated process and measurement noise were proposed. A new filter for the case of colored measurement noise was derived that has several advantages over previous filters. In addition, filters for the out-of-sequence measurement problem were derived that do not depend on the white noise assumption, but incorporate the influence of colored and correlated process and measurement noise. The performance of different colored noise filters was evaluated in a Monte Carlo simulation.

**REFERENCES**


