Abstract—A crucial point in the decision-level identity fusion is to combine information in an appropriate way to generate an optimal decision, according to the individual information coming from a set of different sensors. An interesting approach was developed for the decision-level identity fusion, which use optimization techniques to minimize an objective function which measure the dissimilarities between the combination result and the set of initial sensor reports. Several objective functions were already proposed for the Similar Sensor Fusion (SSF) and the Dissimilar Sensor Fusion (DSF) models. In this paper, we present these fusion methods, we raise some questions and make some improvements, and finally we study the behaviour of these fusion rules on several examples.

Keywords: Probability theory, combination rules, Similar Sensor Fusion, Dissimilar Sensor Fusion

I. INTRODUCTION

The aim of a fusion process is to provide a result which best fit the information provided by a set of sensors. A distinction should however be made between two different situations: the fusion of information provided by similar sensors and the fusion of information provided by dissimilar sensors. From this point of view, two different fusion models can be defined:

• the Similar Sensor Fusion (SSF) model: is used to fuse information from similar sensors, providing reports on common characteristics. This fusion model can also be applied to fuse distinct reports from the same sensor, acquired at different moments. It should not decrease the uncertainty in one or more elements from the frame of discernment, but it should only help to eliminate inconsistencies among reports. The sensors can only confirm each other’s reports and the goal of this fusion model is to find a result which is most consistent with all the sensor reports.

• the Dissimilar Sensor Fusion (DSF) model: is used to fuse information from dissimilar and independent sensors, providing reports on different characteristics. Reports from these sensors can reinforce each other to decrease the uncertainty in one or more elements from the frame of discernment. The goal of this fusion model is to find a result which best represents the increased certainty in the elements of the frame of discernment, according to all the sensor reports.

More details about these two fusion models can be found in [1], and the authors show through some examples the goals and the mechanisms of each of these fusion models.

Based on this classification, each combination rule developed in the last years for probability theory, evidence theory, possibility theory or fuzzy set theory, should clearly be categorized for the SSF model or for the DSF model.

In [1], [2] three combination rules for probability theory are presented. These three combination rules were developed according to an optimization process. Moreover, the technique used to developed the three combination rules can be extended to an entire class of combination rules based on optimization techniques. First two techniques were developed for the SSF model, and the third technique was developed for the DSF model.

In this paper we wish to make an analysis (through some examples) of these three fusion methods developed using some optimization techniques.

This paper is structured as follows. Section II draws a list of the possible sensor reports. In Section III we present the three combination rules developed in [1], [2]. Some examples are studied in Section IV. Section V is the conclusion.
II. SENSOR REPORT FORMS

Let $\Omega = \{a_1, \ldots, a_N\}$ denote the set of $N$ exhaustive and mutually exclusive hypotheses or objects. Let’s consider each sensor report as a collection of subsets of $\Omega$ and the associated belief or likelihood values. Let also consider a degree of reliability associated to each sensor report, denoted as Degree of Confidence (DoC), which is a non-negative number. In [1], the author presents several particular representations of the sensor reports.

1) probability report :

$$D = \left\{ \text{DoC} = w, \right. \left. P(a_i) = r^i \in [0, 1], \forall i = 1, \ldots, N \right\} \quad (1)$$

where $\sum_{i=1}^{N} r^i = 1$. Usually, DoC = $w \in [0, 1]$, but it is possible to consider $\text{DoC}_{\text{max}} = \infty$. Here, $r^i$ represents the probability of the singleton $a_i$.

2) ratio type report :

$$D = \left\{ \text{DoC} = w = r^1 + \ldots + r^L, \right. \left. P(\omega_i^1) : \ldots : P(\omega_i^L) = r^1 : \ldots : r^L \right\} \quad (2)$$

where $\omega_i^\ell \subseteq \Omega$ and $r^\ell \geq 0, \forall \ell = 1, \ldots, L$. Thus, $\text{DoC}_{\text{max}} = \infty$. Here, $r^\ell$ represents the likelihood value between the subsets $\omega_i^\ell$.

Note that a ratio type report can be seen as a partial knowledge of a probability type report. From a probability type report one can compute an infinity of equivalent ratio type reports, which incorporate the whole knowledge of the probability type report. Inversely, from a ratio type report, one can compute an infinity of probability type reports, which are not necessarily equivalent.

3) partition type report

$$D = \left\{ \text{DoC} = w = r^1 + \ldots + r^L, \right. \left. P(\omega^1) : \ldots : P(\omega^L) = r^1 : \ldots : r^L \right\} \quad (3)$$

where $\omega^i \subseteq \Omega$ and $r^\ell \geq 0, \forall \ell = 1, \ldots, L$ with $\omega^i \cap \omega^j = \emptyset$ for $i \neq j$, and $\omega^1 \cup \ldots \cup \omega^L = \Omega$. Thus, $\text{DoC}_{\text{max}} = \infty$. Here, $r^\ell$ represents the likelihood value between the subsets $\omega^\ell$.

Note that the partition type report is a special case of the ratio type report. From probabilistic point of view, the comparisons expressed in the ratio-type report and partition-type report can be expressed in various forms using intersections and unions of the $\omega^\ell$’s without changing the probabilities of the original subsets. Such a comparison can also be broken into multiple comparisons. For example, $P_1(a_1) : P_1(a_2) : P_1(a_3) = 1 : 1 : 1$ can be expressed as $P_1(a_1) : P_1(a_2 \vee a_3) : P_1(a_3) = 1 : 2 : 1$. Although the probabilistic interpretation is the same, an optimal representation should be used when computing the fusion result. In [1], the author considers the optimal representation as the one having no redundancy in it, but there is no constraint saying that all the elements from the frame of discernment should be part of the report. In this situation, the sensor reports $D_1$ and $D_2$

$$D_1 = \left\{ w = 1, \right. \left. P_1(a_1) : P_1(a_3) = 0.9 : 0.1 \right\}$$

$$D_2 = \left\{ w = 1, \right. \left. P_2(a_1) : P_2(a_2) : P_2(a_3) = 0.9 : 0 : 1 \right\}$$

should not be considered as different sensor reports if the two are defined over the same frame of discernment.

We propose then to define the optimal representation for the sensor reports to be the one with no redundancy in it, and moreover, all the elements from the frame of discernment have to be part of the report even if some null likelihood values have to be added to the report representation. For example, let $\Omega = \{a_1, a_2, a_3\}$ be a three element frame of discernment. The sensor report $D_1$ is not an optimal representation and should be replaced by the sensor report $D_2$.

III. COMBINATION RULES FOR PROBABILITY THEORY BASED ON OPTIMIZATION PROCESSES

In this section we present three combination rules for probability theory, developed using some optimization techniques which [1], [2]. Two combination rules were designed for the SSF model and one for the DSF model.

First, we introduce the general principle for the three combination rules. Let $D_k, k = 1, \ldots, K$ be a set of sensors reports to combine and let $D_f$ be the fused report. The fusion result is a probability distribution given by the set of probabilities $P_f(a_i) = p_i, \forall i = 1, \ldots, N$ such that $\sum_{i=1}^{N} p_i = 1$, which best fits the knowledge from the given sensor reports. A global cost function $C_{fK}$ is defined to measure the discrepancy between the fusion result and the $K$ sensor reports.

The three combination rules have to minimize the global cost function $C_{fK}(P)$, according to the constraints imposed by a final probability function $p_i \geq 0, \forall i = 1, \ldots, N$ and $\sum_{i=1}^{N} p_i = 1$. The DoC of the fused information is also computed. These combination rules are in fact constrained optimization problems which can be solved by existent interior point techniques given in [3] and [4].

The fusion methods can operate in both sequential fusion mode and batch fusion mode. However, the two fusion modes are providing different results. The complexity of the two fusion modes (batch and sequential) is identical since the optimization problem is not dependent of the number of sensors to be fused, but only of the
cardinality of the frame of discernment. Thus, when the order of the reports to be fused is not important, and
the same weight should be assign to each report, the
batch mode has to be considered. For the combination
of streaming data, the last report is more important than
the first reports, and the sequential fusion mode should
be considered.

\[ \sum \] is obtained by minimizing the global cost function from
fusion result at all. In other words, no part of
\[ w \] where
\[ C \] the fusion of
\[ D \] all the confidence associated with
\[ k \]cies among reports. Thus, a set of partial cost functions
between the ideal fusion result
\[ C \] where
\[ D \] is the value of the cost function \( C_k(P) \) evaluated using the probability assignments of the fusion
result \( D_f \). \( C_{k\text{max}} \) is given by

\[ C_{k\text{max}} = \max_{i=1,\ldots,N} c_k(P)^2 \sum_{\ell=1}^{L_k} \left( \frac{r^\ell}{w_k} - \frac{P_f(\omega_k^\ell)}{c_k(P)} \right)^2 \]

When all the \( K \) sensors assign probabilities to all the
singletons of \( \Omega \), each sensor report \( D_k \) can be written
as a probability sensor report and it is straightforward to
show that :

\[ P_f(a_i) = \sum_{k=1}^{K} \frac{(w_k)^2 r^i_k}{\sum_{k=1}^{K} (w_k)^2}, \quad i = 1, \ldots, N \]

Thus, the optimal fusion result is obtained by a
weighted average of the sensor reports using \((w_k)^2\) as
weighting coefficients.

One drawback of the CQ fusion method defined
previously, is that the results are dependent of which
representation for the sensor reports is used. The cost
function from Equation (4) is dependent of the represen-
tation of the sensor reports. We propose to use the
CQ fusion method only with the optimal representation
of the sensor reports described in Section II. Example 1
shows the differences between the results obtained using
the various representations of sensor reports.

2) Combination rule based on Kullback-Leibler’s dis-
tance: The combination rule based on Kullback -
Leibler’s distance was introduced first in [1]. We present
here this fusion method which we denote as the KL
fusion method. Consider \( K \) sensors each exploring some
common characteristics of an object and providing
ratio type reports \( (D_k) \). Let \( C_k(P) \) be a set of convex
quadratic cost functions, given by :

\[ C_k(P) = c_k(P)^2 \sum_{\ell=1}^{L_k} \left( \frac{r^\ell}{w_k} - \frac{P_f(\omega_k^\ell)}{c_k(P)} \right)^2 \]

where
\[ c_k(P) = P_f(\omega_k^1) + \cdots + P_f(\omega_k^{L_k}) \]
and let the global cost function be :

\[ C_{f_k}(P) = \sum_{k=1}^{K} (w_k)^2 C_k(P) \]

Thus, the fusion result using the CQ fusion method
is obtained by minimizing the global cost function from
Equation (5), using the constraint given by a probability
distribution \( \sum_{i=1}^{N} p_i = 1 \), with \( p_i \geq 0, \forall i \).

The DoC of the fusion result is given in [1] as follows:

\[ w_f = \sum_{k=1}^{K} w_k \cdot \frac{C_{k\text{max}} - C_k(D_f)}{C_{k\text{max}}} \]

The DoC associated with the fusion result \( D_f \) is

\[ \sum \]
defined as follows:

\[ w_f = \sum_{k=1}^{K} w_k e^{-C_k(D_f)} \]  

(10)

It can be shown that the expression given by Equation (10) as defined in [1], cannot always be computed, particularly, in situations when a null likelihood value \( r_k^i \) is associated to a subset \( \omega_k^i \) in the \( D_k \) sensor report, and the fused result associates also a null probability to the subset \( \omega_k^i \). Equation (10) should be reviewed so it can be used in any situation. In a sequential process, if the DoC value cannot be updated, the global cost function given in Equation (8) cannot be computed for future steps, and thus, the KL fusion method becomes useless (see Example 3 from Section IV).

In the special case where all the \( K \) sensors assign probabilities to all the singletons of \( \Omega \), each sensor report \( D_k \) can be written as a probability sensor report as in Equation (1) and it is straightforward to show that:

\[ P_f(a_i) = \sum_{k=1}^{K} w_k r_k^i, \quad i = 1, \ldots, N \]  

(11)

Thus, the optimal fusion result is obtained by a weighted average of the sensor reports using \( w_k \) as weighting coefficients.

Note that the solution provided by the CQ and KL fusion methods is not always unique. In this situation, the fusion result can be represented as a partition type report.

B. Combination rule based on the analytic center for the DSF model (reviewed)

The aim of the DSF model is to best represent the increased certainty in the elements of the frame of discernment, according to all the sensor reports. Consider \( K \) independent sensors observing an object. Each sensor explores some different characteristics of the object and sends its own identity declaration in a sensor report to the fusion center. A global cost function, defined in [1], involves only the probability distributions of each sensor and the final probability distribution.

\[ C_{fK}(P) = \sum_{k=1}^{K} w_k \sum_{i=1}^{N} \frac{1}{r_k^i} p_i - \sum_{i=1}^{N} \ln p_i \]  

(12)

One drawback of the formulation proposed in [1] is that it cannot be used if the probability distributions to be combine have null probabilities associated to some singletons (\( r_k^i = 0 \) for some \( k \) and \( i \)). We review this formulation for the analytic center fusion method and we propose the global cost function:

\[ C_{fK}(P) = \sum_{k=1}^{K} w_k \sum_{i=1}^{N} \frac{1}{r_k^i} p_i - \sum_{i=1}^{N} \ln p_i \]  

(13)

The fusion solution is obtained solving the convex optimization problem which is to minimize Equation (13), using the constraint given by a probability distribution \( \sum_{i=1}^{N} p_i = 1 \), with \( p_i \geq 0 \), \( \forall i \). This problem is a special case of the analytic center problem in linear programming presented in [4]. We denote this fusion method as AC. The DoC \( w_{fK} \) associated with fusion result \( D_{fK} \) is defined as:

\[ w_{fK} = 1 - \prod_{k=1}^{K} (1 - w_k) \]  

(14)

The analytic center fusion method can operate in both the sequential and batch fusion modes.

IV. EXAMPLES

In this section we consider some simple examples to test the fusion methods (CQ, KL and AC) presented in Section III and we analyze the results. A comparison with the basic average (AV) is made for the SSF fusion methods. Ratio type reports and partition type reports are first transformed into probability type reports, using the pignistic transformation from [5], before using the AV fusion method.

The first example shows the differences between the results obtained using the optimal representation and non-optimal representations for the sensor reports.

**Example 1** Let \( \Omega = \{ a_1, a_2, a_3 \} \) be a frame of discernment with three elements and let \( D_1 \) and \( D_2 \) be two conflictual but similar and independent sensor reports (example from [6]).

\[ D_1 = \left\{ \begin{array}{c} w = 1, \\ P_1(a_1) : P_1(a_3) = 0.9 : 0.1 \end{array} \right\} \]

\[ D_2 = \left\{ \begin{array}{c} w = 1, \\ P_2(a_2) : P_2(a_3) = 0.9 : 0.1 \end{array} \right\} \]

Let denote by \( D_1^* \) and \( D_2^* \) the optimal representations of \( D_1 \) and \( D_2 \) respectively in the frame of discernment \( \Omega \). Table I shows the results when the CQ combination rule is performed between the sensor reports \( D_1 \) (or \( D_1^* \)) and \( D_2 \) (or \( D_2^* \)). We can remark that the CQ fusion method is performing differently, according to the type of representation for each sensor report. A comparison with the simple average is also presented.

We remark that not only the results are different according to the type of representation used in the
combination process, but the conclusions can change radically.

<table>
<thead>
<tr>
<th>Combination</th>
<th>$P_I(a_1)$</th>
<th>$P_I(a_2)$</th>
<th>$P_I(a_3)$</th>
<th>DoC</th>
</tr>
</thead>
<tbody>
<tr>
<td>CQ($D_1,D_2$)</td>
<td>0.4737</td>
<td>0.4737</td>
<td>0.0526</td>
<td>2</td>
</tr>
<tr>
<td>CQ($D_1^*,D_2$)</td>
<td>0.9217</td>
<td>0.0298</td>
<td>0.0485</td>
<td>1.29</td>
</tr>
<tr>
<td>CQ($D_1^<em>,D_2^</em>$)</td>
<td>0.0298</td>
<td>0.9217</td>
<td>0.0485</td>
<td>1.29</td>
</tr>
<tr>
<td>CQ($D_1^<em>,D_2^</em>$)</td>
<td>0.4500</td>
<td>0.4500</td>
<td>0.1000</td>
<td>1.33</td>
</tr>
<tr>
<td>KL</td>
<td>0.4500</td>
<td>0.4500</td>
<td>0.1000</td>
<td>NaN</td>
</tr>
<tr>
<td>AV</td>
<td>0.4500</td>
<td>0.4500</td>
<td>0.1000</td>
<td>N/A</td>
</tr>
</tbody>
</table>

### Table I
**Example 1. Combination using different representations of sensor reports**

Take for instance that when combining the sensor report $D_1$ under its optimal representation $D_1^*$ with the sensor report $D_2$, the result is providing singleton $a_1$ with higher probability than provided in $D_1$ even if $a_1$ is absent in the sensor report $D_2$. This is a counter-intuitive behaviour for the SSF fusion model, since there should be no reinforcement in the singleton beliefs. We can remark that when combining the optimal representations, a simple average is performed (as stated in Section III-A.1). So, to avoid any counter-intuitive behaviours, the solution is to use the optimal representation proposed in Section II. We notice that the KL and AV fusion methods provide the same results as the CQ fusion method applied to the optimal representation of the sensor reports. However, the KL fusion rule cannot compute the DoC value (see discussion in Section III-A.2).

In the next example we show the capability for the CQ and KL combination rules to extract the real information from partial probability distributions.

### Example 2
Let $\Omega = \{a_1, a_2, a_3\}$. Suppose $D_1$, $D_2$ and $D_3$ are three similar and independent partition type sensor reports:

- $D_1 = \{w_1 = 1, P_1(a_1 \lor a_2) : P_1(a_3) = 0.8 : 0.2 \}$
- $D_2 = \{w_2 = 1, P_2(a_1) : P_2(a_2 \lor a_3) = 0.2 : 0.8 \}$
- $D_3 = \{w_3 = 1, P_3(a_1 \lor a_3) : P_3(a_2) = 0.4 : 0.6 \}$

Note that $D_1$, $D_2$ and $D_3$ are partial knowledges of the same probability distribution $P_0(a_1) = 0.2$, $P_0(a_2) = 0.6$ and $P_0(a_3) = 0.2$. Since the three sensor reports are provided by similar sensors, we combine them using the rules developed for the SSF model. A combination rule for the SSF model should be able to fuse the partial knowledge from the three sensor reports and find the real probability distribution. Table II compares the results obtained using the combination rules CQ, KL and AV.

<table>
<thead>
<tr>
<th>Combination</th>
<th>$P_I(a_1)$</th>
<th>$P_I(a_2)$</th>
<th>$P_I(a_3)$</th>
<th>DoC</th>
</tr>
</thead>
<tbody>
<tr>
<td>CQ</td>
<td>0.2000</td>
<td>0.6000</td>
<td>0.2000</td>
<td>3</td>
</tr>
<tr>
<td>KL</td>
<td>0.2000</td>
<td>0.6000</td>
<td>0.2000</td>
<td>3</td>
</tr>
<tr>
<td>AV</td>
<td>0.2667</td>
<td>0.4666</td>
<td>0.2667</td>
<td>N/A</td>
</tr>
</tbody>
</table>

### Table II
**Example 2 : Comparison for the SSF model**

We can remark that the result provided by the CQ and KL combination rules recovers the real probability distribution, which is not the case for the AV fusion rule.

We should notice that the CQ and KL combination rules find the probability distribution which minimizes the distance between itself and the initial probability distributions. In this example, the techniques based on the optimization process (CQ and KL) perform well, and the results are satisfactory.

In the following example we show how the CQ and KL combination techniques can have a counter-intuitive behaviour in a sequential fusion mode.

### Example 3
Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose $D_1$, $D_2$, ..., $D_5$ are five similar and independent ratio type reports:

- $D_1 = \{w_1 = 1, P_1(a_1 \lor a_2) : P_1(a_3 \lor a_4 \lor a_5) = 0.6 : 0.4 \}$
- $D_2 = \{w_2 = 1, P_2(a_1 \lor a_4) : P_2(a_2) : P_2(a_3 \lor a_5) = 1 \cdot 0.4 : 0.2 : 0.4 \}$
- $D_3 = \{w_3 = 1, P_3(a_1 \lor a_2 \lor a_3) : P_3(a_3 \lor a_5) = 0.6 : 0.4 \}$
- $D_4 = \{w_4 = 1, P_4(a_1 \lor a_4) : P_4(a_2 \lor a_3) = 0.7 : 0.3 \}$
- $D_5 = \{w_5 = 1, P_5(a_1 \lor a_2 \lor a_3) : P_5(a_5) = 0.5 : 0.5 \}$

Note that $D_1$, $D_2$ and $D_3$ are partition type reports, which is not the case for $D_4$ and $D_5$. Moreover, the sensor reports $D_1$, $D_2$ and $D_3$ are presented by an optimal representation as defined in Section II, while the sensor reports $D_4$ and $D_5$ are not in the optimal representation. Let $D_1^*$ and $D_5^*$ be the optimal representations of the sensor reports $D_4$ and $D_5$ respectively. $D_4^*$ and $D_5^*$ are partition type reports and will be used with the KL fusion method, instead of $D_4$ and $D_5$ respectively. We consider that these reports do not arrive at the fusion center in the
same time: $D_1$ arrives first and $D_5$ arrives last. In this example, we study a sequential process, but at each step we combine all available sensor reports in a batch mode, which is equivalent to the fusion of streaming data.

First, we study the CQ combination rule in a sequential process using the batch mode, when the non optimal representations $D_1$ and $D_5$ are used. The results are presented in Table III.

The probability distribution $P_f$ is the best approximation of the two partial knowledges given by the sensor reports $D_1$ and $D_2$. We notice that the fusion result gives a null probability for singletons $a_4$ and $a_5$ even if the probability of these two singletons were never null in the initial sensor reports. This result is contrary to our intuition. When the first three sensor reports are combined, the fusion result keeps a null probability for singletons $a_4$ and $a_5$ even if the probability of these two singletons were never null in the initial sensor reports. This result is also contrary to our intuition. Next, we realize the combination of the first four sensor reports. We remark that the probability assign to singletons $a_1$ and $a_4$ in $D_4$ becomes greater than their probabilities from $D_f$. Their probabilities in $D_4$ should be greater than those in $D_f$, but it is not the case for singleton $a_1$.

Moreover, even if the singleton $a_5$ has no probability in $D_f$, and there is no probability assign to it in $D_4$, its probability is updated in $D_f$ and it increases even more than the probability of singleton $a_4$, to which the sensor report $D_4$ assign a non null probability. Finally, we realize the combination of all five sensor reports and the results ($P_f$). The probability of the singleton $a_4$ is increasing from $P_4$ to $P_5$, even if in the sensor report $D_5$ the singleton $a_4$ has a null probability. Moreover, singleton $a_4$ becomes more probable than singleton $a_1$ which was the favourite until now, and which had a non-null probability in the $D_5$ sensor report.

We can conclude to a completely non-intuitive behaviour of the CQ fusion method when non optimal representations for the sensor reports are used in a sequential fusion process.

Next, let consider the combination of the five sensor reports with optimal representations ($D_1$, $D_2$, $D_3$, $D_4$, and $D_5$), in a sequential process, using the batch fusion mode. The CQ, KL and AV fusion methods are analyzed and Table IV shows the result for this example. Some counter-intuitive behaviours observed when fusing the $D_1$ to $D_5$ sensor reports with the CQ fusion method vanished or decreased. However, we notice that between

<table>
<thead>
<tr>
<th>Information to be combined</th>
<th>$P_f(a_1)$</th>
<th>$P_f(a_2)$</th>
<th>$P_f(a_3)$</th>
<th>$P_f(a_4)$</th>
<th>$P_f(a_5)$</th>
<th>DoC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1 \oplus D_2$</td>
<td>0.4000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>2</td>
</tr>
<tr>
<td>$D_1 \oplus D_2 \oplus D_3$</td>
<td>0.4000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>3</td>
</tr>
<tr>
<td>$D_1 \oplus D_2 \oplus D_3 \oplus D_4$</td>
<td>0.3970</td>
<td>0.1396</td>
<td>0.2129</td>
<td>0.1237</td>
<td>0.1268</td>
<td>3.93</td>
</tr>
<tr>
<td>$D_1 \oplus D_2 \oplus D_3 \oplus D_4 \oplus D_5$</td>
<td>0.2246</td>
<td>0.2112</td>
<td>0.0353</td>
<td>0.2481</td>
<td>0.2808</td>
<td>4.83</td>
</tr>
</tbody>
</table>

**TABLE III**

**Example 3. SSF fusion model : Sequential process using the batch mode and the CQ combination rule.**

<table>
<thead>
<tr>
<th>Information to be combined</th>
<th>Fusion method</th>
<th>$P_f(a_1)$</th>
<th>$P_f(a_2)$</th>
<th>$P_f(a_3)$</th>
<th>$P_f(a_4)$</th>
<th>$P_f(a_5)$</th>
<th>DoC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1 \oplus D_2$</td>
<td>batch CQ</td>
<td>0.4000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>batch KL</td>
<td>0.4000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>AV</td>
<td>0.2500</td>
<td>0.2500</td>
<td>0.1667</td>
<td>0.1667</td>
<td>0.1667</td>
<td>N/A</td>
</tr>
<tr>
<td>$D_1 \oplus D_2 \oplus D_3$</td>
<td>batch CQ</td>
<td>0.4000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>batch KL</td>
<td>0.4000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>AV</td>
<td>0.2333</td>
<td>0.2333</td>
<td>0.1778</td>
<td>0.1778</td>
<td>0.1778</td>
<td>N/A</td>
</tr>
<tr>
<td>$D_1 \oplus D_2 \oplus D_3 \oplus D_4$</td>
<td>batch CQ</td>
<td>0.4553</td>
<td>0.1211</td>
<td>0.2737</td>
<td>0.1026</td>
<td>0.0474</td>
<td>3.94</td>
</tr>
<tr>
<td></td>
<td>batch KL</td>
<td>0.4500</td>
<td>0.1500</td>
<td>0.3000</td>
<td>0.1000</td>
<td>0.0000</td>
<td>NaN</td>
</tr>
<tr>
<td></td>
<td>AV</td>
<td>0.2625</td>
<td>0.2125</td>
<td>0.1708</td>
<td>0.2209</td>
<td>0.1333</td>
<td>N/A</td>
</tr>
<tr>
<td>$D_1 \oplus D_2 \oplus D_3 \oplus D_4 \oplus D_5$</td>
<td>batch CQ</td>
<td>0.3595</td>
<td>0.1397</td>
<td>0.1428</td>
<td>0.1516</td>
<td>0.2065</td>
<td>4.73</td>
</tr>
<tr>
<td></td>
<td>batch KL</td>
<td>0.4225</td>
<td>0.1259</td>
<td>0.2178</td>
<td>0.0243</td>
<td>0.2095</td>
<td>NaN</td>
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<tr>
<td></td>
<td>AV</td>
<td>0.2433</td>
<td>0.2033</td>
<td>0.1700</td>
<td>0.1767</td>
<td>0.2067</td>
<td>N/A</td>
</tr>
</tbody>
</table>

**TABLE IV**

**Example 3. SSF fusion model : Combination of information in a sequential process using the batch mode**
the steps 2 and 3, there is an increasing value of the probability of \( a_5 \) even if \( a_5 \) has a null probability in the sensor report \( D_1^* \). The DoC value for the KL fusion method cannot be computed after the 3rd fusion step.

When the sequential fusion mode is used to fuse the \( D_1 \) to \( D_5 \) sensor reports in the optimal representation, the KL fusion method cannot be used. During the combination of \( (D_1 \oplus D_2) \oplus D_3 \) the DoC value \( w_{13} \) cannot be computed, which blocks the sequential fusion process. The results obtained using the CQ fusion method are presented in Table V.

In conclusion of this example, the CQ fusion method has a counter-intuitive behaviour in such a sequential process, when the combination is realized by the batch fusion mode at each fusion step. We notice that the use of the optimal representation for the sensor reports reduces the counter-intuitive behaviour, but does not eliminate it. The KL fusion method has a more intuitive behaviour than the CQ fusion method. However, when fusing only the first three sensor reports, the probability distribution obtained using the KL combination rule is identical to the one obtained using the CQ combination rule, and both provide some counter-intuitive results \( (P_f(a_4) = P_f(a_5) = 0 \text{ even if the initial sensor reports were not against } a_4 \text{ and } a_5) \). We do not recommend the use of these fusion methods for the characterization of a sequential fusion process.

In the following example we show how the AC fusion method acts in a high conflicting situation.

**Example 4** Let \( \Omega = \{a_1, a_2, a_3\} \) be a frame of discernment with three elements and let \( D_1 \) and \( D_2 \) be two dissimilar and independent sensor reports, in conflict, which are given in Example 1. The AC combination rule for the DSF model is supposed to work with probability sensor reports, as stated in Section III-B. The AC combination rule, as defined first in [1], cannot be used to combine the \( D_1 \) and \( D_2 \) sensor reports, because they are not defined by complete probability distributions. It cannot be used neither to combine the \( D_1^* \) and \( D_2^* \) representations of the sensor reports \( D_1 \) and \( D_2 \) respectively because some null probabilities are assigned. However, with the new formulation and restriction given in Equation (13), the result obtained using the AC combination rule is given in Table VI. Note that this result is very close to the one given by the CQ combination rule developed for the SSF model, which is a normal behaviour for the combination of high conflicting information.

**Example 5** Consider the following independent and dissimilar sensor reports:

\[
D_1 = \left\{ w_1 = 1 \right\} \\
D_2 = \left\{ w_2 = 1 \right\} \\
D_2' = \left\{ w_3 = 1 \right\}
\]

Note that \( D_2 \) and \( D_2' \) sensor reports are very close. According to the continuity principle, when \( D_1 \) is combined with \( D_2 \) and \( D_2' \) respectively, the results should be close too. Table VII shows the fusion results when the AC fusion method is used to combine these sensor reports.

### Table V

<table>
<thead>
<tr>
<th>Information to be combined</th>
<th>( P_f(a_1) )</th>
<th>( P_f(a_2) )</th>
<th>( P_f(a_3) )</th>
<th>( P_f(a_4) )</th>
<th>( P_f(a_5) )</th>
<th>DoC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (D_1 \oplus D_2) \oplus D_3 )</td>
<td>0.4000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>3</td>
</tr>
<tr>
<td>( ((D_1 \oplus D_2) \oplus D_3) \oplus D_4^* )</td>
<td>0.4273</td>
<td>0.1727</td>
<td>0.3727</td>
<td>0.0273</td>
<td>0.0000</td>
<td>3.91</td>
</tr>
<tr>
<td>( (((D_1 \oplus D_2) \oplus D_3) \oplus D_4^<em>) \oplus D_5^</em> )</td>
<td>0.4107</td>
<td>0.1561</td>
<td>0.3561</td>
<td>0.0360</td>
<td>0.0410</td>
<td>4.69</td>
</tr>
</tbody>
</table>

**Example 4. Reviewed AC combination rule for the DSF fusion model**

**Example 5. AC combination rule for the DSF fusion model**

We notice that the two results are very different. Thus, we conclude that the continuity property is not satisfied and that the AC fusion method is very sensitive to changes, especially when the sensor reports to be fused are in conflict. Moreover, when the Bayesian inference is...
used, Singleton $a_1$ is considered as the best probable in both situation, which is not the case when combining $D_1$ with $D_2$ using the AC fusion method. We showed here a counter-intuitive behaviour of the AC fusion method.

V. CONCLUSION

In this paper we presented an interesting approach to combine ratio type and partition type reports and in a more general way probability type reports. A clear distinction is made between the Similar Sensors Fusion model and Dissimilar Sensors Fusion model and the goals for each model is presented. We analyze three combination rules for probability theory which use optimization techniques. Two combination rules are based on a convex quadratic optimization and on the Kullback-Lieber’s distance respectively, which were developed for the SSF model, and a combination rule is based on the analytic center problem, which was developed for the DSF model. One of the strengths of the two fusion methods for the SSF model is that they can work with partial probability distributions. These fusion methods are capable to recover a probability distribution from multiple partial knowledges of this distribution. However, as shown through some examples, the CQ and KL combination rules fail to provide intuitive behaviours in a sequential fusion process. We also proved that the AC combination rule is very sensitive to small changes, and thus do not respect the continuity principle. We showed that these three fusion methods are not well-characterized in some particular situations and we tried to improve the fusion methods in some of these situations. Future works should try to improve the fusion methods presented here, try other cost functions which could be more appropriate according to particular applications, and analyze more complex scenario-tests to prove the utility or the uselessness of the fusion methods based on optimization techniques in the identification problems.

REFERENCES