Abstract - In a preceding paper at Fusion2009, the existence and characterization of a unique decomposition of the joint conditional density of the states of two targets has been proven. This decomposition consists of a weighted sum of a permutation invariant density and a permutation strictly variant density. In the current paper we exploit this unique decomposition for the development of a novel particle filter for tracking two closely spaced linear Gaussian targets. Thanks to the unique decomposition this novel particle filter is able to provide a conditional estimate of the track swap probability. The remarkable working of this novel particle filter is demonstrated through running Monte Carlo simulations for an example in tracking two closely spaced targets.

Keywords: Bayesian estimation, Multitarget tracking, Track Swap, Particle filtering, Unique decomposition.

1 Introduction

The development of the Sampling Importance Resampling (SIR) particle filter [1]-[3] has created a complete novel approach towards approximating an exact Bayesian filter arbitrarily close, and has led to the development of a large variety of particle filters (e.g. [4]-[6]) that typically outperform established approaches in maintaining single target tracks. The success of particle filtering for single target tracking strongly motivated particle filter developments for multiple target tracking (e.g. [7-9]). In spite of all these developments, particle filtering for closely spaced targets still poses unresolved problems [10-14].

For a two target scenario the basic problem is depicted in Figure 1. Two targets, one labelled blue and one labelled red, start out well separated. Then they move towards each other, continue close to each other for some period and then separate again. Initially, a joint particle filter starts with only one type of labelling, which is shown by the initial sub-cloud of circles for the red target and the initial sub-cloud of squares for the blue target. However, once the targets start to move close to each other, at a certain point the sub-clouds have mixed circle and square labels. This label mixing is not caused by some sub-optimal behaviour of the particle filter, though represents well the behaviour of an exact Bayes filter. Even when the particle filter was initially certain about the identity of the particle labels, it starts to allow label permutations when the targets paths move for a while close to each other.

Figure 1. Mixing of particle labels and MMSE state estimation impact.

The large red and blue dots in Figure 1 show the MMSE state estimates at the end of the scenario; they are somewhere in between the two particle clouds. In target tracking, this is an undesired type of behavior which is known as track coalescence. For particle filtering some alternative output forms have been developed which do not suffer from track coalescence. [9] develops a particle clustering prior to MMSE estimation per cluster. [13-14] develop MAP estimation approaches. A disadvantage of both approaches is that the tracking output may switch in an uncontrolled way between permutation options [14]. Motivated by this finding, [15] develops a unique decomposition of the joint conditional density $p_{x,y}(x)$ of two targets in a sum of a permutation invariant density $p_{x,y}^I(x)$ and a strictly permutation variant density $p_{x,y}^\chi(x)$:
\[ p_{x_{i,t}}(x) = \alpha_i p_{x_{i,t}}^0(x) + (1 - \alpha_i) p_{x_{j,t}}^0(x) \]

where \( \alpha_i \) is the conditional probability of permutation invariance.

The aim of this paper is to develop a particle filter that allows to estimate the joint conditional density of two linear Gaussian targets in this decomposed way, including estimation of the permutation invariance probability \( \alpha_i \).

The established way of reasoning in multi target tracking is that when each target behaves Gaussian, then the joint conditional density is characterized by a Gaussian mixture. However, in the sequel it will become clear that even in tracking two closely spaced Gaussian targets permutation invariance causes non-Gaussian effects playing a key role.

The paper is organized as follows. Section 2 formulates the track maintenance problem for two targets. Section 3 summarizes the unique decomposition results of [15]. Section 4 develops the basis for particle filtering using this unique decomposition. Section 5 develops the novel particle filter. Section 6 provides simulation results for an example of two closely spaced targets. Section 7 draws conclusions.

2 Track maintenance problem

We consider two targets and assume that the state of each target is modelled as a linear Gaussian system:

\[ x_{i,t} = a_i x_{i,t-1} + w_{i,t}, \quad i = 1, 2 \]  

where \( x_{i,j} \) is the \( n \)-vectorial state of the \( i \)-th target, \( a_i \) is an \( (n \times n) \)-matrix and \( \{w_{i,t}\} \) is a sequence of independent identically distributed (i.i.d.) zero mean Gaussian variables of dimension \( n' \), with \( E\{w_{i,t}w_{i,t}^T\} = W_i \), \( \text{Det}(W_i) > 0 \), and \( \{w_{i,t}\} \) and \( \{w_{j,t}\} \) mutually independent, and also independent of \( x_{i,0} \) and \( x_{2,0} \).

We assume that a potential measurement originating from target \( i \) is modelled as a linear system:

\[ z_{i,t} = h_i x_{i,t} + g_i v_{i,t}, \quad i = 1, 2 \]

where \( z_{i,t} \) is an \( m \)-vector, \( h_i \) is an \( (m \times n) \)-matrix and \( g_i \) is an \( (m \times m') \)-matrix, and \( \{v_{i,t}\} \) is a sequence of i.i.d. standard Gaussian variables of dimension \( m' \), with \( \{v_{i,t}\} \) and \( \{v_{j,t}\} \) mutually independent. Moreover \( \{v_{i,t}\} \) is independent of \( x_{i,0} \) and \( \{w_{j,t}\} \) for all \( i, j \).

Stacking target states and potential measurements yields:

\[ x_i = Ax_{i-1} + w_i \]
\[ z_i = Hx_i + Gv_i \]

with: \( x_i = \text{Col}\{x_{i,t}, x_{i,t'}\} \), \( z_i = \text{Col}\{z_{i,t}, z_{i,t'}\} \), \( w_i = \text{Col}\{w_{i,t}, w_{i,t'}\} \), \( v_i = \text{Col}\{v_{i,t}, v_{i,t'}\} \), \( A = \text{Diag}\{a_i, a_j\} \), \( W = \text{Diag}(W_i, W_j) \), \( H = \text{Diag}(h_i, h_j) \), \( G = \text{Diag}(g_i, g_j) \).

where \( \text{Col}\{y_1, y_2\} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) and \( \text{Diag}\{y_1, y_2\} = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \).

At moment \( t = 1, 2, ..., T \), a vector observation \( y_i \) is made of the two targets. The relation between \( y_i \) and \( x_i \) satisfies:

\[ \mathcal{X} y_i = z_i = Hx_i + Gv_i \]  

where \( \mathcal{X} \triangleq \mathcal{X}_i \otimes I \), with \( I \) a unit-matrix (of size \( m \)), \( \otimes \) Kronecker product, and \( \{\mathcal{X}_i\} \) a sequence of i.i.d. \( 2 \times 2 \) permutation matrices, which is independent of \( \{x_i, v_i, w_i\} \).

For two targets, \( \mathcal{X}_i \) either is the \( 2 \times 2 \) unity matrix \( I \), or its permutation \( \Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

The multi-target track maintenance problem considered is to estimate \( x_i \) in a recursive way from observations \( Y_i = \{y_i; 0 \leq s \leq t\} \), where \( y_0 = \{} \). In spite of all linear Gaussian assumptions, the permutation matrix process \( \{\mathcal{X}_i\} \) in (5) makes this is a non-linear filtering problem. A recursive characterization of the exact joint conditional density is given in [Blom&Bloem, 2006, Theorem 1] for a more general problem setting. For the specific problem considered here, the joint conditional density \( p_{x_{i,t}}(\cdot) \), of the joint state \( x_i \) given \( Y_i \), satisfies the recursion:

\[ p_{x_{i,t}}(x) = \frac{1}{c_i} \sum_{\mathcal{X}_i \in \{I, \Pi\}} N(\mathcal{X} y_i; Hx, GG^T) \cdot \int_{\mathbb{R}^m} N(x; Ax', W'). p_{w_{i,t}}(x') dx' \]

with \( c_i \), a normalization constant. Because of the summation over \( \mathcal{X} \in \{I, \Pi\} \), \( p_{x_{i,t}}(\cdot) \) is a mixture of \( 2^t \) Gaussians, provided that the initial density \( p_{x_{i,0}}(\cdot) \) is Gaussian.
3 Unique decomposition

This section exploits the unique decomposition theory developed in [15].

**Definition 1:** We say that the conditional density of the joint two-target state is permutation invariant at moment \( t \) if for all \( x \in \mathbb{R}^{2n} \)

\[
p_{s,v_{t}}(x) = p_{s,v_{t}}(\Pi x)
\]

(6)

**Remark 1:** The standard way of working in target tracking is to use the (global) MMSE state estimation for the target tracking output, which satisfies:

\[
\hat{x}_{t}^{MMSE} = \int_{\mathbb{R}^{2n}} x p_{s,v_{t}}(x) dx
\]

(7)

By using (6), (7) yields:

\[
\hat{x}_{t}^{MMSE} = \int_{\mathbb{R}^{2n}} x \left[ p_{s,v_{t}}(x) + p_{s,v_{t}}(\Pi x) \right] dx / 2
\]

\[
= \left[ x_{t}^{MMSE} + \Pi x_{t}^{MMSE} \right] / 2
\]

(8)

which implies \( x_{t}^{MMSE} = \Pi x_{t}^{MMSE} \). This proofs that when two targets have a permutation invariant joint conditional density, they also have equal MMSE estimated states.

**Remark 2:** If the joint conditional density \( p_{s,v_{t}}(x) \) is permutation invariant at moment \( t = \tau \), then it remains permutation invariant for all \( t > \tau \) (see [15], Theorem 1).

When there is an \( x \in \mathbb{R}^{2n} \) for which

\[
p_{s,v_{t}}(x) \neq p_{s,v_{t}}(\Pi x)
\]

(9)

then we know from Definition 1 that for this \( x \), \( p_{s,v_{t}}(x) \) is not permutation invariant, i.e. \( p_{s,v_{t}}(x) \) is permutation variant. In order to capture such permutation variance in an explicit way, we adopt a kind of permutation variant definition which applies for all \( x \in \mathbb{R}^{2n} \).

**Definition 2:** We say that the conditional density \( p_{s,v_{t}}(x) \) of the joint two-target hybrid state is strictly permutation variant if for all \( x \in \mathbb{R}^{2n} \):

\[
p_{s,v_{t}}(x).p_{s,v_{t}}(\Pi x) = 0
\]

(10)

**Remark 3:** It can be verified that the following holds true:

\[
p_{s,v_{t}}(x).p_{s,v_{t}}(\Pi x) = 0
\]

if and only if \( p_{s,v_{t}}(x) = 0 \) and/or \( p_{s,v_{t}}(\Pi x) = 0 \).

**Theorem 1** ([15], Theorem 2)

\( p_{s,v_{t}}(x) \) admits a unique decomposition in a weighted sum of a permutation invariant density \( p_{s,v_{t}}^{\chi}(x) \) and a strictly permutation variant density \( p_{s,v_{t}}^{\Pi}(x) \), i.e. for all \( x \in \mathbb{R}^{2n} \):

\[
p_{s,v_{t}}(x) = \alpha p_{s,v_{t}}^{\chi}(x) + (1-\alpha) p_{s,v_{t}}^{\Pi}(x)
\]

(12)

\[
p_{s,v_{t}}^{\chi}(x) = p_{s,v_{t}}^{\chi}(\Pi x)
\]

(13a)

\[
p_{s,v_{t}}^{\Pi}(x).p_{s,v_{t}}^{\Pi}(\Pi x) = 0
\]

(13b)

where \( 0 \leq \alpha \leq 1 \).

[15] also characterizes the terms appearing in the unique decomposition of the joint conditional density as follows.

**Theorem 2** ([15], Theorem 3)

In the unique decomposition of Theorem 1, the weight \( \alpha \) and the permutation invariant density \( p_{s,v_{t}}^{\chi}(x) \) satisfy:

\[
\alpha = \int_{\mathbb{R}^{2n}} \min\{p_{s,v_{t}}(x), p_{s,v_{t}}(\Pi x)\} dx
\]

(14)

\[
p_{s,v_{t}}^{\chi}(x) = \min\{p_{s,v_{t}}(x), p_{s,v_{t}}(\Pi x)\} / \alpha, \text{ if } \alpha > 0
\]

(15)

\[
p_{s,v_{t}}^{\Pi}(x) = \left[ p_{s,v_{t}}(x) - \alpha p_{s,v_{t}}^{\chi}(x) \right] / (1-\alpha), \text{ if } \alpha < 1
\]

(16)

**Example:** What is the effect of the decomposition on a joint Gaussian density \( \mathcal{N}(\mu, \Sigma) \), with \( \mu = \text{Col}[4,5] \) and \( \Sigma = \text{Diag}[2,2] \) ? A contour plot of this Gaussian density is given in Figure 2a. Decomposition of this joint Gaussian density according to Theorem 2 yields the permutation invariant part in Figure 2b, and the strictly permutation variant part in Figure 2c. Numerical evaluation of the weight yields \( \alpha = 0.62 \), which means that this joint Gaussian density is around 62% permutation invariant. This percentage rapidly decreases when the centre of the joint Gaussian is moved away from the \( \epsilon_{1} = \epsilon_{2} \) line.
4 Decomposed particles

In preparation to characterizing equations for the particles, we first characterize the recursion of the decomposed joint conditional density as follows.

**Theorem 3**

The joint conditional density $p_{x|y_i}(x)$ satisfies:

\[
p_{x|y_i}(x) = c_i \sum_{\chi} \frac{1}{4}(1-\alpha_{i-1})N\{x|y_i\}; Hx,GG^T\} p_{x|y_{i-1}}(x) +
\]

\[
+ c_i \sum_{\chi} \frac{1}{4}\alpha_{i-1}N\{x|y_i\}; Hx,GG^T\} \sum_{\chi'} p_{x|y_{i-1}}(x') \quad (17)
\]

with:

\[
p_{x|y_{i-1}}^X(x) = \int_{\mathbb{R}^n} N\{x; Ax^*,W\} p_{x|y_{i-1}}^X(x')dx' \quad (18)
\]

\[
p_{x|y_{i-1}}^X(x) = \int_{\mathbb{R}^n} N\{x; Ax^*,W\} p_{x|y_{i-1}}^X(x')dx' \quad (19)
\]

Proof: Omitted, because of space limitation.

Theorem 3 shows how measurement updating combines novel permutation possibilities to the permutation possibilities of the predicted conditional density.

Next we show how these novel permutation possibilities can be folded in the unique decomposition of $p_{x|y_i}(x)$.

For this we assume that $p_{x|y_i}(x)$ is an empirical density which is spanned by the following set of $N_p$ decomposed particles: \[\{x_{i-1}^j \in \mathbb{R}^n; \mu_{i-1}^j, \mu_{i-1}^X \in [0,1]; j = 1, ..., N_p\}\] such that for all $j \in \{1, ..., N_p\}$, $\mu_{i-1}^j + \mu_{i-1}^X = 1/ N_p$, and $\text{P}$-almost surely $x_{i-1}^j \neq x_{i-1}^j \neq \Pi x_{i-1}^j$ for all $i \neq j$. These decomposed particles span the empirical density

\[
\hat{p}_{x|y_{i-1}}(x) = \sum_{j=1}^{N_p} \mu_{i-1}^X \delta(x - x_{i-1}^j) +
\]

\[
+ \sum_{j=1}^{N_p} \frac{1}{2} \mu_{i-1}^X [\delta(x - x_{i-1}^j) + \delta(x - \Pi x_{i-1}^j)]
\]

This can also be written as:

\[
\hat{p}_{x|y_{i-1}}(x) = \alpha_{i-1} \hat{p}_{x|y_{i-1}}^X(x) + (1 - \alpha_{i-1}) \hat{p}_{x|y_{i-1}}^X(x)
\]

with: $\alpha_{i-1} = \sum_{j=1}^{N_p} \mu_{i-1}^X$.
\[ \hat{p}_{x_{i-1},j}^x(x) = \sum_{j=1}^{N} \frac{1}{\alpha_{i-1}} \mu_{x_{i-1}}^{x,j} \sum_{x \in I, i} \delta(x - x_{i}^{j}) / \alpha_{i-1} \]

\[ \hat{p}_{x_{i-1}}^x(x) = \sum_{j=1}^{N} \frac{1}{\alpha_{i-1}} \mu_{x_{i-1}}^{x,j} \delta(x - x_{i}^{j}) / (1 - \alpha_{i-1}) \]

Plugging this into (18)-(19) and evaluation yields:

\[ \hat{p}_{x_{i-1},j}^x(x) = \sum_{j=1}^{N} \frac{1}{\alpha_{i-1}} \mu_{x_{i-1}}^{x,j} N\{x; x_{i}^{j}, W\} / \alpha_{i-1} \]

\[ \hat{p}_{x_{i-1}}^x(x) = \sum_{j=1}^{N} \frac{1}{\alpha_{i-1}} \mu_{x_{i-1}}^{x,j} N\{x; x_{i}^{j}, W\} / (1 - \alpha_{i-1}) \]

with: \( x_{i}^{j} = Ax_{i-1}^{j} \). Next, plugging the above into (17) and taking into account that:

\[ N\{\Pi x; \Pi x_{i}^{j}, W\} = N\{x; x_{i}^{j}, W\} \]

\[ N\{x; \Pi x_{i}^{j}, W\} = N\{x; \Pi x_{i}^{j}, W\} \]

we get:

\[ \hat{p}_{x_{i-1}}(x) = \frac{c_i}{2} \sum_{x} N\{x_{i}^{j}; Hx, GG^T\} \sum_{j=1}^{N} \mu_{x_{i-1}}^{x,j} N\{x; x_{i}^{j}, W\} \]

\[ + \frac{c_i}{4} \sum_{x} N\{x_{i}^{j}; Hx, GG^T\} \sum_{j=1}^{N} \mu_{x_{i-1}}^{x,j} N\{x; x_{i}^{j}, W\} \]

The next step is to evaluate the products of Gaussian densities in these sums. Using Multiple Model measurement update equations and subsequent evaluation, yields:

\[ \hat{p}_{x_{i-1}}(x) = \frac{c_i}{2} \sum_{x} \sum_{j=1}^{N} \mu_{x_{i-1}}^{x,j} N\{x_{i}^{j}; Hx_{i}^{j}, Q\} N\{x; x_{i}^{j}, W\} \]

\[ + \frac{c_i}{2} \sum_{x} \sum_{j=1}^{N} \mu_{x_{i-1}}^{x,j} N\{x_{i}^{j}; Hx_{i}^{j}, Q\} N\{x; x_{i}^{j}, W\} \]

where: \( x_{i}^{j} = 0 + K(x_{i}^{j} - Hx_{i}^{j}) \)

\( K = WH^T Q^{-1} \)

\[ Q = HHWT + GG^T \]

\[ P = W - WHW \]

Because the terms \( N\{x_{i}^{j}; Hx_{i}^{j}, Q\} \) are \( x \)-invariant, these multiply with the weights, i.e.

\[ \beta_{x_{i-1}}^{x,j}(x) = \mu_{x_{i-1}}^{x,j} N\{x_{i}^{j}; Hx_{i}^{j}, Q\} / c_i \]

\[ \beta_{x_{i-1}}^{x,j}(x) = \mu_{x_{i-1}}^{x,j} N\{x_{i}^{j}; Hx_{i}^{j}, Q\} / c_i \]

with \( c_i \) such that:

\[ \sum_{x} \sum_{j=1}^{N} [\beta_{x_{i-1}}^{x,j}(x) + \beta_{x_{i-1}}^{x,j}(x)] = 1 \]

Hence, \( \hat{p}_{x_{i-1}}(x) \) can be written as:

\[ \hat{p}_{x_{i-1}}(x) = \sum_{x} \beta_{x_{i-1}}^{x,j}(x) N\{x; x_{i}^{j}(x), P\} \]

\[ + \sum_{x} \sum_{j=1}^{N} \frac{1}{\alpha_{i-1}} \beta_{x_{i-1}}^{x,j}(x) N\{x; x_{i}^{j}(x), P\} \]

With the above result, we have shown that by starting with an empirical density \( \hat{p}_{x_{i-1}}(x) \) which is spanned by \( N_p \) decomposed particles, we get a density \( \hat{p}_{x_{i-1}}(x) \) which is a sum of \( 4N_p \) Gaussians, all having the same covariance \( P \).

### 5 Decomposed particle filtering

In this section we will show how \( \hat{p}_{x_{i-1}}(x) \) satisfying (20) is sampled by \( N_p \) new decomposed particles which span an empirical density approximation of \( \hat{p}_{x_{i-1}}(x) \). This leads to the decomposed particle filter cycle steps in Table 1.

The evolution and correction steps are based on the equations derived in section 4. From section 4 we also know that after one filter cycle, \( \hat{p}_{x_{i-1}}(x) \) is a sum of \( 4N_p \) Gaussians, each of which having the same covariance \( P \). Straightforward application of the unique decomposition characterization of Theorem 2 to this Gaussian sum would lead to a very complicated exercise. Instead, we perform this characterization such that the unique decomposition applies at the locations of the newly sampled particles.

**Resampling:** In preparation to drawing the \( N_p \) new samples, we first draw independent \( N_p \) pointers \( \{k^j_i, \chi^j_i\} \) to the \( 4N_p \) Gaussians. Subsequently we draw \( \chi^j_i \) from \( N\{x; \hat{x}_{i}^{j}(\chi^j_i), P\} \) for \( i = 1, ..., N_p \). Because \( Det(W) > 0 \) we have \( Det(P) > 0 \), which implies that almost surely \( x_{i}^{j} \neq x_{i}^{j} \neq \Pi x_{i}^{j} \) for all \( i \neq j, i, j = 1, ..., N_p \).

**Unique decomposition:** The next step is to perform the characterization of Theorem 2, which means that the weight \( 1/ \) \( N_p \) is decomposed in the permutation invariant and strictly variant weights \( \mu_{x_{i-1}}^{x,j}(\chi) = \rho_{i}^{j} \) (with \( 0 < \rho_{i}^{j} < 1 \)) and \( \mu_{x_{i-1}}^{x,j}(\chi) = (1 - \rho_{i}^{j})N_p^{-1} \). The derivation of the permutation invariant fraction \( \rho_{i}^{j} \) is given in an Appendix.

**Optimal Track Continuity Assignment output:** The target tracking output objective is to produce an estimated track state output which does not suffer from track coalescence, and neither leads to switching between permutation versions. Elaboration of this objective leads to the Decomposed Particle Filter Track State output in Table 2. Because of space limitations this elaboration is here omitted.
For $j = 1, ..., N_p$:

- Evolution: $\bar{x}_i^j = A x_{i-1}^j$
- Correction:
  \[
  \beta_{X,i}^j(\chi) = \frac{\mu_{X,i}^j}{N} \left[ \chi_{j,i}; H \bar{x}_i^j, Q \right] / c_i \\
  \hat{\beta}_{X,i}^j(\chi) = \frac{\mu_{X,i}^j}{N} \left[ \chi_{j,i}; H \bar{x}_i^j, Q \right] / c_i
  \]
  with $Q = HWHT + GG^T$ and $c_i$ such that
  \[
  \sum_{j=1}^N \beta_{X,i}^j(\chi) + \hat{\beta}_{X,i}^j(\chi) = 1
  \]

Resampling:

For $i = 1, ..., N_p$, draw samples, with replacement:

\[
\left( \chi_i^j, \bar{x}_i^j \right) \sim \tilde{p}_{\chi,\bar{x}}(j, \chi) \triangleq \beta_{X,i}^j(\chi) + \hat{\beta}_{X,i}^j(\chi) \\
\bar{x}_i^j \sim N\left( \chi_i^j, K(\chi_i^j) \right) \\
K = WHW^T + GG^T \\
P = W - KH
\]

The new decomposed weights become:

\[
\mu_i^{X,j} = \hat{\rho}_i^j / N_p \\
\mu_i^{X,j} = (1 - \hat{\rho}_i^j) / N_p
\]

with:

\[
\hat{\rho}_i^j = \frac{\sigma_i^2(x_i^j)}{\sigma_i^2(x_i^j) + \sigma_i^2(\hat{x}_i^j)} \\
\sigma_i^2(x_i^j) = \sum_{\chi} \sum_{j=1}^N \beta_{X,i}^{X,j}(\chi) N \left[ \chi; \bar{x}_i^j(\chi), P \right] \\
\sigma_i^2(\hat{x}_i^j) = \sum_{\chi} \sum_{j=1}^N \hat{\beta}_{X,i}^{X,j}(\chi) N \left[ \chi; \hat{x}_i^j(\chi), P \right]
\]

The new decomposed particles are:

\[
\left\{ x_i^j \in \mathbb{R}^{2n}, \mu_i^{X,j}, \mu_i^{X,j} \in [0,1]; \; i = 1, ..., N_p \right\}
\]

These particles span the empirical density:

\[
\hat{p}_{\chi,\bar{x}}(x) = \frac{1}{N_p} \sum_{i=1}^{N_p} \mu_i^{X,j} \mathbf{1}(x - x_i^j) + \frac{1}{N_p} \sum_{j=1}^{N_p} \hat{\mu}_i^{X,j} \mathbf{1}(x - \bar{x}_i^j)
\]

Produce track output (see Table 2) and perform next cycle.

The equations in Table 2 form an extended version of the the approach developed in [18] in assigning estimated new track states to the previous track output states. In [18] it is also explained that this may lead to an unknown track swap impact. In contrast to [18], our decomposed particle filtering cycle allows us to estimate track swap probability, as a consequence of which we always know the conditional track swap probability $\hat{p}_{\text{Swap}, t}^{\text{TCA}}$.

### 6 Monte Carlo simulation

In this section Monte Carlo simulations are conducted for the novel decomposed particle filter using $N_p = 100$ joint particles. The two target tracking scenario is from [17]. The underlying target model evolves according to discretized continuous white-noise acceleration [18], and target position is observed in noise, i.e. for the coefficients in (1)-(2) holds:

\[
a_j = \begin{bmatrix} \frac{1}{T_s} & 0 \\ 0 & 1 \end{bmatrix}, \quad W_i = \sigma_a \begin{bmatrix} \frac{1}{2} T_s & 0 \\ 0 & \frac{1}{2} T_s \end{bmatrix}, \quad h_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad g_i = \sigma_m
\]

where $\sigma_a$ represents the standard deviation of acceleration noise and $\sigma_m$ represents the standard deviation of the measurement error. Monte Carlo simulations containing 100 runs are performed over a period of 40s. The initial track estimates are accurate. The scenario parameter values are $d = 100m$, $V_{\text{initial}} = 75m/s$, $T_s = 1s$, $\sigma_m = 30m$ and $\sigma_a = 50m^2/s$.

Results of the Monte Carlo simulations are presented in Figures 3-6. Figure 3 shows typical results for one run by the novel decomposed particle filter. The TCA track position output stays pretty close to the true targets (Fig. 3a), in spite of significant track swap probability (Fig. 3b). Figure 4 shows what happens when the novel decomposed particle filter is used to provide MMSE output: then track coalescence behaviour is back again, which is very well in line with expectation. Figure 5 shows the results of a
normal SIR PF using ten thousand particles. Figure 6 shows the RMS of OSPA errors [19] in position for the novel decomposed particle filter with TCA output, with MMSE output, and a normal SIR Particle Filter with MMSE output.

![Decomposed PF TCA Estimates](image)

3a. Novel PF results (● = measurement, x and + = track output)

![Decomposed PF TCA Swap probability](image)

3b. Estimated track swap probability

Figure 3. Track output of novel decomposed particle filter.

![Decomposed PF MMSE Estimates](image)

Figure 4. MMSE track output of the novel decomposed particle filter shows track coalescence again.

![OSPA Position RMSE](image)

Figure 6. RMS of OSPA errors in position for the novel decomposed particle filter with TCA and MMSE outputs, and for a SIR particle filter with MMSE output.

7 Concluding remarks

For the problem of maintaining tracks of two maneuvering targets from unidentified observations, we have developed a novel particle filter. Explicit use has been made of the unique decomposition by [15] of a conditional density for the joint state of two targets into a weighted sum of a permutation invariant density and a strictly variant density. The novel particle filter uses decomposed particles, and is named “decomposed particle filter”. From a target tracking output perspective, this decomposed particle filter has two unique capabilities:

- It provides track output that behaves continuous and stays remarkably close to the observations;
- It provides an estimate of the probability that the presented tracks are swapped regarding the true target locations.

Through MC simulations for a simple example of two targets that start well separated, then fly towards each other,
and then separate again, the novel particle filter’s unique capabilities have been demonstrated to work well.

Directions for follow up research are to extend the novel particle filter for covering missed detections and false measurements, to extend it to hybrid and non-linear targets, to cover limited sensor resolution, and to cover more than two targets.

References


8 Appendix

From Theorem 2, for all \( i = 1,\ldots,N_p \):

\[
p^x_{i|\mathbf{z}}(\mathbf{x}_i') = \frac{\min_x \{ p_{x|\mathbf{z}}(\mathbf{x}_i') \} / \alpha_i}{\sum_x \{ p_{x|\mathbf{z}}(\mathbf{x}_i') \} / \alpha_i} \quad (21)
\]

From (20) we get:

\[
p_{x|\mathbf{z}}(\mathbf{x}_i') = \frac{1}{2} \alpha_i \sigma^x_i(\mathbf{x}_i') + \sigma^x_i(\mathbf{x}_i') \quad (22)
\]

with:

\[
\sigma^x_i(x) = \sum_{\mathbf{z}} \sum_{\chi} \sum_{\chi'} \beta_i(x_i') N(x; \chi) \sigma_i(\mathbf{x}_i', P_i) \quad (23)
\]

\[
\sigma^x_i(x) = \sum_{\mathbf{z}} \sum_{\chi} \sum_{\chi'} \beta_i(x_i') N(x; \chi) \sigma_i(\mathbf{x}_i', P_i) \quad (24)
\]

Substituting (22) in (21) yields:

\[
\alpha_i p^x_{i|\mathbf{z}}(\mathbf{x}_i') = \min_x \{ \frac{1}{2} \alpha_i \sigma^x_i(\mathbf{x}_i') + \sigma^x_i(\mathbf{x}_i') \} = \frac{1}{2} \alpha_i \sigma^x_i(\mathbf{x}_i') + \min_x \{ \sigma^x_i(\mathbf{x}_i') \} \quad (25)
\]

The permutation invariant fraction \( \rho_i' \) equals:

\[
\rho_i' = \frac{\alpha_i p^x_{i|\mathbf{z}}(\mathbf{x}_i') + \frac{1}{2} \sigma^x_i(\mathbf{x}_i')}{\sigma^x_i(\mathbf{x}_i') + \sigma^x_i(\mathbf{x}_i')} \quad (26)
\]

Substituting (25) into this, yields:

\[
\rho_i' = \frac{\sigma^x_i(\mathbf{x}_i') + \min_x \{ \sigma^x_i(\mathbf{x}_i') \}}{\sigma^x_i(\mathbf{x}_i') + \sigma^x_i(\mathbf{x}_i')} \quad (26)
\]

Equations (23), (24) and (26) are used to evaluate the novel decomposed weights in Table 1.