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Abstract—In this paper we study the problem of determining the optimal spatial node density for deployment of a Wireless Sensor Network (WSN) for detection of a randomly located target in a sensing field. We formulate an optimization problem for the single-hop scenario and account for factors such as the Medium Access Control (MAC) protocol that is used, the wireless channel's propagation characteristics, a randomized sleep/wake-up scheduling protocol, network coverage constraints, the energy consumed, the time to reach a decision, and the total number of nodes available. We show that the optimal node density that minimizes the average Decision Error Probability (DEP) at the CH is a function of the network parameters such as average wake-up rate, propagation path-loss exponent and the lifetime of the network. Simulations are used to study the many optimal trade-offs that are possible.

Keywords: Distributed Detection, Wireless Sensor Network Design.

I. INTRODUCTION

We consider a scenario in which a Wireless Sensor Network (WSN) is deployed to perform target detection in a sensing field. We assume that the target is stationary and that each node performs a local binary hypothesis test. Within each cluster the local decisions from the sensor nodes are transmitted to the CH that fuses the received information to produce a global decision about the event under consideration. Furthermore to conserve energy, we assume that the nodes follow a randomized sleep wake-up scheduling protocol. A network designer is typically interested in key system performance metrics such as network lifetime, coverage, time required to collect data from all the nodes in the network, and the Quality of Monitoring (QoM) of the stochastic events that occur or are being sensed in the environment. In the case of target detection applications a relevant QoM metric would be the overall Decision Error Probability (DEP). In general, the node deployment density and the scheduling protocols can directly affect the QoM performance metrics. Knowing a priori how many sensor nodes to deploy in a region for a particular sensing application is thus crucial from a network design point of view. Our goal is to determine the optimal values of the wake-up scheduling protocol and the spatial node densities that minimize the overall DEP, taking into account factors such as the underlying MAC protocol, faulty sensors, and signal propagation characteristics. We formulate a general optimization framework to solve the problem and study interesting tradeoffs between the lifetime of the network, the achievable DEP, and the number of sensor nodes deployed. The rest of the paper is organized as follows; in Section II we review the most relevant work in the area of WSN deployment for distributed detection applications. Section III we describe our system model and assumptions followed by Section IV, in which we formulate and solve the optimization problem. In Section V we provide numerical simulation results and finally we conclude in Section VI.

II. RELATED WORK

Most of the previous works that have studied a similar problem have not taken into account all the important factors that can affect the performance of the QoM metric. In [1] the authors study the problem of designing a WSN to detect the location of a target moving in a discretized spatial domain. By solving a Mixed Integer Programming (MIP) problem, they determine the optimal deterministic CH placement strategy that would maximize the error exponents. One drawback of their approach is that a deterministic placement strategy maybe not be possible to implement for large scale networks. In [2] the authors study the error exponents for detecting a Gauss-Markov signal using a sensor network. They consider the problem in a 1-D setting and determine the node density that maximizes the error exponent for both randomly placed and fixed, equally-spaced sensors. A similar setting is considered in [3], in which the authors find the optimal node density for energy constrained random networks using a Gauss-Markov random field model in a 2-D setting. In terms of sensor deployment for target detection, the authors in [4] consider the problem of deploying sensors to detect a moving target. They propose a deployment strategy that incorporates a sensor collaboration framework which maximizes the probability of detection of the target along any path in the sensing field. None of the above papers consider the effect of an underlying MAC scheduling protocol or account for signal propagation characteristics and noisy sensor observations. The authors in [5] study the problem of determining the optimal deterministic node deployment strategy for the distributed detection of an exponentially randomly located target with a known mean. The optimal inter-node spacing that minimizes the Bhattacharyya bound on the error probability of the Bayesian detector at the CH is obtained for 1 and 2 D networks. As with some of the previous papers, the effect of the MAC protocols has not been
analyzed and, as mentioned before deterministic deployment may not be practical for very large scale networks. The authors in [6] propose a data-dependent control policy for transmission over the slotted ALOHA MAC protocol to improve the performance of distributed detection applications. They derive the optimal transmission probability for each sensor mote such that the overall DEP at the CH is minimized. The authors also extend their framework to the multi-cluster scenario. Our approach differs from their’s on two fronts. First, we explicitly account for the probability of transmission in each slot by considering the node density and the wake-up rate for each node. Second, in our framework we take into consideration the communication-channel errors and measurement errors that can occur, for example because of defective sensor nodes. Finally, the motivation of our work is to determine the optimal spatial node density, rather than the channel access probability for each sensor.

III. SYSTEM MODEL

A. Assumptions and Network Topology

The sensors are assumed to be randomly and uniformly deployed with a spatial node density of $\lambda$ nodes/unit area over a sensing field to monitor the presence of a stationary target. Once deployed, they organize into a single-hop, clustered network using a distributed clustering algorithm, such as the one proposed in [7]. In the case that the distribution of the sensor nodes is governed by a Homogeneous Poisson Process (HPP) the optimal node density would correspond to the intensity of the HPP. We make the following additional assumptions about our system:

(i) The clusters are circular shaped with radius of $R$ and the cluster head (CH) located at the center. All of the sensors transmit on the same frequency and the underlying MAC protocol is the slotted ALOHA random access protocol. Each slot is of length $\ell$ seconds, which corresponds to the time taken to transmit a single packet from a sensor to the CH.

(ii) All sensors transmit at the same power level ($P_{tx}$) and we define the time period over which the CH collects data from the sensors as a data collection cycle. Each data collection cycle $m$, consists of $K$ slots and there are a total of $M$ data collection cycles. At the end of each data collection cycle, the CH fuses the local decisions to produce a global decision. Fig. 1 shows the format and the structure of the data collection protocol.

B. Event Detection Model

The sensors in the cluster perform binary event detection between two hypotheses. The local observation at sensor $k$ can be expressed as:

$$H_0: y_k(t) = w_k(t)$$
$$H_1: y_k(t) = \theta_k + w_k(t)$$

where $k = 1, \ldots, N_T$, $w_k(t)$ is i.i.d with distribution $\sim N(0, \sigma_n^2)$, $t = 1, \ldots, T$ is the sampling time interval and $\theta_k$ is the signal to be detected. The length of the sampling time is assumed to be a fraction of a time slot; i.e., $T = \eta \ell$, where $0 \leq \eta \leq 1$. In typical target detection applications, the sensor nodes perform the local detection by measuring the energy of the transmitted signal from the target source. Due to the propagation characteristics of most signals, the energy decays as a function of the distance from the source to the sensor mote. Hence $\theta_k$ is a function of the energy of the signal transmitted by the target, which is denoted by $E_{tx}$, and the distance between the target and sensor mote $k$ is denoted by $d_k^T$, where the term $tx$ denotes the target. Therefore, the received signal energy (RSE) can be denoted by $\theta_k$ and is given by:

$$\theta_k = E_{tx} F(d_k^T)$$

where $F(d_k^T)$ is a decreasing function with respect to $d_k^T$. For our case, we assume that $F(d_k^T) = \frac{1}{1 + (d_k^T)^\alpha}$, where $\alpha$ is referred to as the path-loss exponent. We consider a decision fusion framework, in which the sensors perform a local Likelihood Ratio Test (LRT) based on their local observations to make a local decision between the two hypothesis. The local decision from sensor $k$ is denoted by $s_k$. The sensor transmits a data packet with $s_k = s_1$ if $H_1$ is decided, or $s_k = s_0$ if $H_0$ is decided. For analytic tractability, we assume without loss of generality that each hypothesis occurs with equal probability: $Pr(H_0) = Pr(H_1) = 0.5$.

The results can easily be generalized to the non-equal prior case. To model the dynamics of the target, we assume that target’s location is also uniformly distributed within the cluster. This implies that $d_k^T$ is an i.i.d random variable for all $k$. Using the results from geometric probability [8] the pdf of $d_k^T$ for $0 < x < 2R$ is given by:

$$f_{d_k^T}(x) = \frac{2x}{R^2} \left[ \frac{2}{x} \cos^{-1} \left( \frac{x}{2R} \right) - \frac{x}{\pi R} \sqrt{1 - \frac{x^2}{4R^2}} \right]$$

Fig. 2. Overall System Model of the WSN. Each sensor makes a binary decision based on the local conditionally independent observation of the stochastic event $y_k(t)$. The decision is transmitted over a slotted ALOHA MAC to the CH, where the information is fused to produce a global decision about the event.
Using (3) the pdf of $\theta_k$ can be derived as:

$$f_{\theta_k}(y) = \frac{f_{d_{k|x}}'(x_1)}{E_{tx}|F'(d_{k|x})|} = \frac{(1 + x_1^{-\alpha})^2 f_{d_{k|x}}'(x_1^T)}{\alpha x_1^{\alpha-1} E_{tx}}$$  \hspace{1cm} (4)

where $x_1 = \frac{E_{tx} }{8} - 1$ and $F'(d_{k|x})$ is the derivative of $F(d_{k|x})$ w.r.t $d_{k|x}$. By the above assumptions, it can be shown that the local probability of detection for sensor node $k$, denoted by $P^k_d = Pr(T_k > \frac{d_k}{T} | \theta_k, H_1)$, and the local probability of a miss, denoted by $P^k_m = Pr(T_k \leq \frac{d_k}{T} | \theta_k, H_0)$, where $T$ is the sufficient statistic for the LRT and is given by the sample mean, i.e. $T_k = \frac{1}{T} \sum_{i=1}^{T} y_k(t)$. Furthermore, from the equal prior assumption it can be shown that the local probability of false alarm is $P^k_f = Pr(T_k > \frac{d_k}{T} | \theta_k, H_0) = P^k_m$ and the local probability of rejection is $P^k_r = Pr(T_k \leq \frac{d_k}{T} | \theta_k, H_0) = P^k_d$. Therefore $Pr(s_k = s_1 | \theta_k, H_1) = P^k_d$ and $Pr(s_k = s_0 | \theta_k, H_0) = P^k_m$. Since the threshold $\frac{d_k}{T}$ depends on the location of the target, which is assumed to be randomly distributed in the field, the expected values of $P^k_d$ and $P^k_m$ are used. These are denoted by $\bar{P}^k_d$ and $\bar{P}^k_m$ and can be calculated as:

$$\bar{P}^k_d = E_{\theta_k}(P^k_d) = \int_{\theta_1}^{\theta_2} \left[1 - Q\left(\frac{y\sqrt{T}}{2\sigma_w}\right)\right] f_{\theta_k}(y) dy$$  \hspace{1cm} (5)

$$\bar{P}^k_m = E_{\theta_k}(P^k_m) = \int_{\theta_1}^{\theta_2} Q\left(\frac{y\sqrt{T}}{2\sigma_w}\right) f_{\theta_k}(y) dy$$  \hspace{1cm} (6)

We observe that since we average over all realizations of $\theta_k$, $\bar{P}^k_d = P_d$ and $\bar{P}^k_m = P_m$. Note that since the target location is uniformly randomly located in the sensing field, the average probability of detection and miss for any node is the same.

C. Packet Generation Model

We assume that the sensors follow an asynchronous sleep/wake up scheduling protocol, wherein each sensor node wakes up independently of other sensors and according to a homogenous Poisson process with mean rate $\Lambda_p$. At the instant that a sensor node wakes up, it senses the phenomenon, makes a local decision on the hypothesis, and transmits the packet to the CH before going back to sleep. Hence, according to this model the data packets are generated according to a Poisson process with a mean rate of $\epsilon \Lambda_p$ packets/slot. Furthermore, to account for defective sensor nodes, we assume that the errors can lead the local hard decisions to flip with probability $\epsilon$. For energy-efficient operation we also assume that the wireless sensors do not use any retransmission control protocol.

D. Packet Success Probability

The underlying MAC protocol will govern the probability of packet success for each sensor in the network. The transmitted data packets from a sensor may collide with transmitted packets from other sensors, due to contention in the medium. The probability of a successful transmission of a packet $s \in \{s_0, s_1\}$ from sensor mote $k$ in each time slot is denoted by $P^{s_i}_{\text{succ}}(k)$ for $i \in \{0, 1\}$. It depends on the number of nodes in the cluster and the average distance of a sensor node to the CH. Having a large number of sensor nodes will lead to a large number of collisions due to the MAC protocol and having the sensor as spaced far from the CH will lead to a higher packet loss rate because of increased signal attenuation. Hence, $P^{s_{\text{succ}}}_{\text{succ}}(k)$ can expressed as:

$$P^{s_{\text{succ}}}_{\text{succ}}(k) = P^{s_{\text{mac}}}_{\text{succ}}(k) P^{s_{\text{dist}}}_{\text{succ}}(k),$$  \hspace{1cm} (7)

where $P^{s_{\text{mac}}}_{\text{succ}}(k)$ represents the probability of successfully transmitting a packet at the MAC layer and $P^{s_{\text{dist}}}_{\text{succ}}(k)$ represents the average probability of successfully decoding a packet transmitted from anywhere within the cluster. Since packets are scheduled to transmit only at the beginning of each time slot, a packet is successfully transmitted if and only if 1 packet is generated in the previous time slot, as depicted in Fig. 1 (b). Let $G_{s_1|H_1}(k)$ and $G_{s_0|H_1}(k)$ denote the number of $s_1$ and $s_0$ packets generated in 1 time slot under $H_1$ by sensor mote $k$. The probability of a successful transmission of a $s_1$ packet under $H_1$ from sensor mote $k$ is thus given as:

$$P^{s_{1|H_1}}_{\text{mac}}(k) = g^1_{s_1|H_1}(k)g^0_{s_0|H_1}(k) \prod_{i=1}^{N} g^0_{s_1|H_1}(i)g^0_{s_0|H_1}(i)$$  \hspace{1cm} (8)

$$= \bar{P}_d + \epsilon (1 - 2 \bar{P}_d) \ell \Lambda_p e^{-\nu \ell \Lambda_p},$$  \hspace{1cm} (9)

where $g^1_{s_1|H_1}(k) = Pr(G_{s_1|H_1}(k) = 1)$, $g^0_{s_0|H_1}(k) = Pr(G_{s_0|H_1}(k) = 0)$, $g^0_{s_1|H_1}(i) = Pr(G_{s_1|H_1}(i) = 0)$ and $g^0_{s_0|H_1}(i) = Pr(G_{s_0|H_1}(i) = 0)$. Using similar arguments and by the symmetry of the problem we have:

$$P^{s_{0|H_1}}_{\text{mac}}(k) = \left(\epsilon + \bar{P}_{m}(1 - 2\epsilon)\right) \ell \Lambda_p e^{-\nu \ell \Lambda_p}$$  \hspace{1cm} (10)

$$P^{s_{0|H_0}}_{\text{mac}}(k) = P^{s_{0|H_1}}_{\text{mac}}(n)$$  \hspace{1cm} (11)

$$P^{s_{0|H_0}}_{\text{mac}}(k) = P^{s_{0|H_1}}_{\text{mac}}(n)$$  \hspace{1cm} (12)

We note that $P^{s_{|H_1}}_{\text{mac}}(k)$ is independent of the type of packet and the hypothesis, as the probability is related to the path-loss due to signal propagation over the transmission distance. We assume a simplified log-distance path-loss model that can model the large scale fading effects such as shadowing. In this case, the received power $P^k_{rx}(\text{dBm})$ from sensor node $k$ is related to the transmit power $P^k_{tx}(\text{dBm})$ of sensor node $k$ via the following relation:

$$P^k_{rx} = \frac{P^k_{tx} - [PL_0 + 10\alpha \log(d_k/d_0)] + X}{\ell}$$  \hspace{1cm} (13)

where $d_k$ is the distance from sensor node $k$ to the CH, $X \sim \mathcal{N}(0, \sigma^2_2)$, $\alpha$ is the path-loss exponent, $PL_0$ is the mean path-loss at a close-in reference distance of $d_0$ near the transmitter. We assume that all nodes transmit with the same transmit power, therefore $P^k_{tx} = P^x_t$. The variance of the Gaussian noise term (measured in dB) is referred to as the dB-spread in the shadowing effect. Typically, in most wireless communication receivers for a packet to be decoded successfully the received signal strength (RSS) is compared against a receiver sensitivity threshold ($\tau_{\text{sen}}$). Only if the RSS is above this threshold value can the packet be decoded.
and the transmission be successful; otherwise, the transmission is considered to be unsuccessful. Therefore:
\[
P^k_{\text{dist}|d_k} = Pr(P^k_{\text{rx}} > \tau_{\text{sen}}) = Q\left(\frac{\tau_{\text{sen}} - \mu(d_k)}{\sigma_x}\right)
\]
where \(Q(.)\) represents the complementary cumulative distribution function (ccdf) of the standard Gaussian r.v and \(\mu(d_k) = P_{\text{tx}} - [P_{\text{Lo}} + 10\alpha \log(d_k/d_0)\]. Since the nodes are deployed uniformly within the circular cluster, it can be shown that
\[
f_{d_k}(r) = \frac{2r}{R^2} \forall 0 \leq r \leq R
\]
Hence we have:
\[
P^s_{\text{dist}} = \int_0^R Q\left(\frac{\tau_{\text{sen}} - \mu(r)}{\sigma_x}\right) \frac{2r}{R^2} dr
\]
Therefore the probability of successfully receiving a packet \(s\) under \(H_i\) is given by:
\[
P^s_{\text{succ}}(k) = P^s_{\text{mac}}(k) P^s_{\text{dist}}
\]
Fig. 2 shows the overall system architecture of the WSN, in which the sensor nodes makes a local binary decision of the stochastic event and transmit their decision over a slotted ALOHA random access channel to the CH. Since the observations are i.i.d across time slots and data collection cycles, w.l.o.g for rest of this paper we study the performance over 1 data collection cycle.

IV. OPTIMAL DEPLOYMENT FORMULATION

We first derive the expressions for the total expected energy consumed in the network and the average DEP achievable at the CH. We then formulate and solve the optimization problem to determine the optimal values for the spatial node density and the mean wake-up rate.

A. Energy Model

To analyze the total energy consumed in this type of network, we assume the standard communication model in which the energy consumed for a single transmission from node \(k\) over a distance of \(d_k\) is given by
\[
E^k_{\text{trans}} = \gamma + \beta(d_k)^\alpha,
\]
where \(\gamma\) is the energy spent in the transmitter electronics circuitry and \(\beta(d_k)^\alpha\) is the energy used in the RF amplifiers to counteract the effect of the transmission path loss. Therefore, the expected energy consumed per sensor node over \(K\) slots is:
\[
E^k_{\text{exp}} = \mathbb{E}[E^k_{\text{trans}}]
\]
\[
= \int_0^R K\ell\Lambda_p(\gamma + \beta r^\alpha) \frac{2r}{R^2} dr
\]
\[
= K\ell\Lambda_p(\gamma + \frac{2K\ell\Lambda_p\beta R^\alpha}{\alpha + 2})
\]
Therefore the total expected energy consumed by the network over \(K\) time slots is given as:
\[
E_{\text{total}} = \sum_{k=1}^{N_T} E^k_{\text{exp}}
\]
\[
= N_T K\ell\Lambda_p(\gamma + \frac{2\beta R^\alpha}{\alpha + 2})
\]

B. Decision Fusion and Probability of Error Analysis

To derive the overall DEP at the CH, let \(N_r^s(k)\) and \(N_0^s(k)\) denote the number of \(s_1\) and \(s_0\) packets successfully received at the CH from sensor mote \(k\) over one data collection cycle. A time slot may be unused because a collision has occurred or no data packets were generated for transmission. We denote the number of unused time slots in a data collection cycle by \(N_{\text{total}}^s\). Furthermore let \(P_{s_0|H_i}(k)\) and \(P_{s_1|H_i}(k)\) denote the probability of successfully receiving an \(s_1\) or \(s_0\) packet, respectively, from sensor mote \(k\) in a time slot under \(H_i\). We then have:
\[
P_{s_1|H_i}(k) = P_{\text{succ}}(k) \frac{P_{\text{dist}}(\gamma + \epsilon(1 - 2P_d))}{P_{\text{dist}}(\gamma + \epsilon(1 - 2P_d))}
\]
Similarly we have:
\[
P_{s_0|H_i}(k) = P_{\text{dist}}(\gamma + \epsilon(1 - 2P_d))
\]
The probability of an unused slot under \(H_i\) is denoted by \(P_{\text{idle}}(H_i)\). We denote the probability of a packet collision under \(H_i\) by \(P_{\text{coll}}(H_i)\) and the probability of an idle slot by \(P_{\text{idle}}(H_i)\). Therefore:
\[
P_{\text{coll}}(H_i) = 1 - P_{\text{idle}}(H_i) e^{-N^r \ell T_p \Lambda_p} - e^{-N^r \ell T_p}
\]
and \(P_{s_0|H_i}(k) = P_{\text{coll}}(H_i) + P_{\text{idle}}(H_i) = 1 - P_{\text{dist}}(\gamma + \epsilon(1 - 2P_d)) e^{-N^r \ell T_p \Lambda_p}\). For ease of notation, we denote the total number of \(s_1\) packets received over 1 data collection cycle by \(N_{s_1}^\text{total}\); similarly, we have \(N_{s_0}^\text{total} = \sum_{k=1}^{N_T} N_{s_0}^s(k)\). Let the received vector of decisions from all nodes over 1 data collection cycle be denoted by \(X = [N_{s_1} N_{s_0} N_{s_u}]^T\), where \([\cdot]^T\) stands for the transpose of the matrix. The pmf of this vector of decisions is given by the following Multinomial distribution under each hypothesis:
\[
f(n_{s_1}, n_{s_0}, n_{s_u}|H_1) = \prod_{k=1}^{N_T} \frac{K! P_{s_1|H_1}^{n_{s_1}} P_{s_0|H_1}^{n_{s_0}} P_{s_u|H_1}^{n_{s_u}}}{n_{s_1}! n_{s_0}! n_{s_u}!}
\]
\[
f(n_{s_1}, n_{s_0}, n_{s_u}|H_0) = \prod_{k=1}^{N_T} \frac{K! P_{s_1|H_0}^{n_{s_1}} P_{s_0|H_0}^{n_{s_0}} P_{s_u|H_0}^{n_{s_u}}}{n_{s_1}! n_{s_0}! n_{s_u}!}
\]
Since the optimal Bayesian decision rule is given by the log-likelihood ratio test (LRT), using the (27) and (28) the log-LRT can be simplified to:
\[
n_{s_1} - n_{s_0} \sim H^1_i 0
\]
We define $Z = (N_{s_1} - N_{s_0})$ as the test statistic that is used by the CH to make the final decision. Defining $q_0 = P_{s_0|H_1}$, $q_1 = P_{s_1|H_1}$, and $q_s = P_{n_s|H_1}$, the exact analytic expression for the average DEP can be expressed as [6]:

$$P_e = 0.5 Pr(Z = 0|H_1) + Pr(Z < 0|H_1)$$

(30)

Defining $X = \{Z = 0|H_1\}$ and $Y = \{Z < 0|H_1\}$, then $Pr(Z = 0|H_1)$ for $K$ even and odd respectively is given as:

$$Pr(X) = 0.5 \sum_{n=0}^{K} \binom{K}{n} (q_0 q_1)^{-n} q_s^{2n}$$

and $Pr(Z < 0|H_1)$ as

$$Pr(Y) = 0.5 \sum_{n=0}^{K-1} \binom{K}{n} (q_0 q_1)^{-n} q_s^{2n-1}$$

Since to evaluate and optimize the DEP using (30) is difficult, we use the Central Limit Theorem (CLT) to approximate the Multinomial distributions for (27) and (28) with a multivariate Gaussian distribution. The numerical results confirm that this is a very good approximation even for a reasonably small number of time slots ($K \approx 20$). The likelihood function of the received vector under each hypothesis can be expressed as:

$$f(\bar{x}|H_1) \sim N(\bar{\mu}_1, \Sigma_1)$$

$$f(\bar{x}|H_0) \sim N(\bar{\mu}_0, \Sigma_0)$$

where $\bar{x} = [n_{s_1}, n_{s_0}, n_{s_u}]$ is the vector of observed decisions, $\bar{\mu}_1 = K[q_1 q_0 q_{s_u}]^T$, $\bar{\mu}_0 = K[q_0 q_1 q_{s_u}]^T$ are the mean vectors and the covariance matrices are:

$$\Sigma_1 = \begin{bmatrix}
K q_1 (1 - q_1) & -K q_1 q_0 & -K q_1 q_{s_u} \\
-K q_1 q_0 & K q_0 (1 - q_0) & -K q_0 q_{s_u} \\
-K q_1 q_{s_u} & -K q_0 q_{s_u} & K q_{s_u} (1 - q_{s_u})
\end{bmatrix}$$

and

$$\Sigma_0 = \begin{bmatrix}
K q_0 (1 - q_1) & -K q_0 q_0 & -K q_0 q_{s_u} \\
-K q_0 q_0 & K q_0 (1 - q_0) & -K q_0 q_{s_u} \\
-K q_0 q_{s_u} & -K q_0 q_{s_u} & K q_{s_u} (1 - q_{s_u})
\end{bmatrix}$$

Note that due to the structure of the covariance matrices we have $|\Sigma_1| = |\Sigma_0|$; therefore, the log-LRT can be simplified to:

$$\ln \frac{f(\bar{x}|H_1)}{f(\bar{x}|H_0)} = (\bar{x} - \bar{\mu}_0)^T \Sigma_0^{-1} (\bar{x} - \bar{\mu}_0) - (\bar{x} - \bar{\mu}_1)^T \Sigma_1^{-1} (\bar{x} - \bar{\mu}_1)$$

(31)

After some tedious algebraic steps, (31) reduces to:

$$n_{s_1} - n_{s_0} \sim H_1 \sim 0$$

(32)

This is exactly the same test statistic as (29); however, since now $(N_{s_1}, N_{s_0})$ are jointly Gaussian, $Z = N_{s_1} - N_{s_0}$ is also Gaussian with a distribution given by

$$f(z|H_1) \sim N(\mu_z|H_1, \sigma_z^2|H_1)$$

$$f(z|H_0) \sim N(\mu_z|H_0, \sigma_z^2|H_0)$$

where $\mu_z|H_1 = K(P_{s_1|H_1} - P_{s_0|H_1})$, $\sigma_z^2|H_1 = KP_{s_1|H_1}(1 - P_{s_1|H_1}) + K P_{s_0|H_1}(1 - P_{s_0|H_1}) + 2K P_{s_1|H_1} P_{s_0|H_1} \sigma_z^2|H_1$. Similarly $\mu_z|H_0 = K(P_{s_0|H_1} - P_{s_1|H_1})$, $\sigma_z^2|H_0 = \sigma_z^2|H_1$. In terms of the expected throughput per slot $S$, we can see that $P_{s_1|H_1} = S(P_d + (1 - 2P_d))$ and $P_{s_1|H_1} = S(P_m + (1 - 2P_m))$. Using these expressions, the average DEP under the Gaussian approximation becomes:

$$P_e = 0.5 Pr(Z > 0|H_0) + 0.5 Pr(Z \leq 0|H_1)$$

$$= Q\left(\sqrt{\frac{K S(P_1 - P_0)^2}{(P_1 + P_0) - S(P_1 - P_0)^2}}\right)$$

(35)

where $Q(.)$ is the complementary cumulative distribution function for a Standard Normal r.v., $P_1 = P_d + (1 - 2P_d)$, $P_0 = P_m + (1 - 2P_m)$ and the expected throughput per slot is given as:

$$S = P^*_{dist} N_T \ell_A \int e^{-N_T \ell_A}$$

(36)

Note that using the Gaussian approximation provides insights into the behavior and dependency of the average DEP on the expected throughput per slot. Intuitively we would expect a higher throughput to lead to a lower DEP, as more local decisions reach the CH.

C. Network Optimization

Our goal is to determine the optimal values for such parameters as the mean wake-up rate of the scheduling protocol $\Lambda_p$ and the node density $\lambda$. In practical deployment scenarios, a network designer can tune these parameters to optimize the overall system performance. We consider the case in which the goal is to minimize the average DEP subject to a minimum network lifetime constraint. Hence we require that after one data collection cycle that the expected energy consumed by the network should be less than some fraction ($\delta$) of the total initial energy of the network. This ensures that the lifetime of the network will be at least $\frac{1}{\delta}$ data collection cycles. Mathematically, this can be expressed as:

$$K N_T \ell_A \left(\gamma + \frac{2\beta R^0}{\alpha + 2}\right) \leq N_T \delta E_0$$

(37)

$$\Lambda_p \leq \frac{\delta E_0}{K C_1}$$

(38)

where $C_1 = \gamma + \frac{2\beta R^0}{\alpha + 2}$. Since in realistic situations a network designer would be able to afford up to a certain number of sensor nodes due to cost/budget constraints, we also impose an upper bound on the total number of sensor nodes that can be deployed in the sensing field. This bound is denoted by $N_{max}$.

The network optimization problem can now be formulated as:

$$\arg \min_{\Lambda_p, N_T} Q\left(\sqrt{\frac{K P^*_{dist} N_T \ell_A e^{-N_T \ell_A} (P_1 - P_0)^2}{(P_1 + P_0) - P^*_{dist} N_T \ell_A e^{-N_T \ell_A} (P_1 - P_0)^2}}\right)$$

(39)

$$0 < \Lambda_p \leq \frac{\delta E_0}{K C_1}$$

(40)

$$0 < N_T \leq N_{max}$$

(41)
If we define the channel load to be $G = N_T \ell \Lambda_p$, the expected throughput per slot can be expressed as $S = \bar{P}_d \lambda e^{-G}$. Substituting the expression for $S$ into the objective function (39) and observing that the $Q(.)$ function is monotonically decreasing, the above optimization problem is equivalent to maximizing the expected throughput per slot subject to the system constraints:

$$
\arg\max_G \bar{P}_d G e^{-G} \quad (42)
$$

$$
0 < G \leq \frac{N_{\text{max}} \delta E_0}{KC_1} \quad (43)
$$

Let the objective function (42) be denoted by $F_1$ and the inequality constraint by $F_2$. Then $F_1$ can be expressed in terms of the expected throughput per slot as:

$$
F_1 = \bar{P}_d G e^{-G} \quad (44)
$$

The gradient of $F_1$ and $F_2$ w.r.t $G$ is given by:

$$
\frac{\partial F_1}{\partial G} = \bar{P}_d e^{-G} (1 - G) \quad (45)
$$

$$
\frac{\partial F_2}{\partial G} = 1 \quad (46)
$$

By the KKT conditions we have:

$$
\frac{\partial F_1}{\partial G} + \mu \frac{\partial F_2}{\partial G} = 0 \quad (47)
$$

$$
\mu (G - \frac{N_{\text{max}} \delta E_0}{KC_1}) = 0 \quad (48)
$$

$$
\mu \geq 0 \quad (49)
$$

Consider two cases for $\mu$: (i) $\mu = 0$ and (ii) $\mu > 0$. We have the following optimal conditions for $G$:

$$
G^* = \begin{cases} 
1 & \text{if } \frac{N_{\text{max}} \delta E_0}{KC_1} \geq 1 \\
\frac{N_{\text{max}} \delta E_0}{KC_1} & \text{if } \frac{N_{\text{max}} \delta E_0}{KC_1} < 1
\end{cases}
$$

We observe that the optimal solution is not unique. Since $G = N_T \ell \Lambda_p$, we can choose any values of $N_T$ and $\Lambda_p$ that satisfy the optimality condition for $G^*$ and constraints (40), (41). From a network designers perspective, the goal would be to determine the number of nodes to deploy in the region. To ensure uniqueness of the optimal solution we thus choose $\Lambda_p^* = \frac{\delta E_0}{KC_1}$ such that constraint (40) is always active. Then we have that:

$$
N_T^* = \begin{cases} 
\frac{1}{\tau} & \text{if } \frac{N_{\text{max}} \delta E_0}{KC_1} \geq 1 \\
\frac{1}{\tau} \left( \frac{N_{\text{max}} \delta E_0}{KC_1} \right) & \text{if } \frac{N_{\text{max}} \delta E_0}{KC_1} < 1
\end{cases}
$$

We now consider a situation in which we do not impose any upper bound on the total number of sensor nodes that can be deployed in the sensing field. In this case, with the condition that $\Lambda_p = \frac{\delta E_0}{KC_1}$, the network optimization problem simplifies to just the unconstrained problem given by:

$$
\arg\max_{N_T} \frac{\bar{P}_d}{KC_1} N_T \ell \delta E_0 e^{-\frac{N_T \ell \delta E_0}{KC_1}} \quad (50)
$$

The optimal solution to (50) is given by $N_T^* = \frac{KC_1 \ell \delta E_0}{\delta E_0} = \frac{1}{\lambda^*}$ which ensures that the optimal channel load $G^*$ is always 1. Once we have found $N_T^*$, the optimal spatial node density follows easily from the relation that $\lambda^* = \frac{N_T^*}{\pi R^2}$.

### V. Numerical Results

For all the numerical results we use the parameter values in Table I. The simulation results are averaged over $10^4$ iterations and the expressions for $\bar{P}_d$ and $\bar{P}_d$ given by (5) and (15) are evaluated numerically using the adaptive Simpson quadrature technique in Matlab. We first study the effect of varying the number of time slots and node density $\lambda$ on the expected throughput per slot and the average DEP. We consider $K = [20 50 100]$ time slots, $\lambda$ is varied from 20 to 1000 nodes/unit area, and the average wake-up rate is set to a constant of $\Lambda_p = 0.8x10^{-6}$.

Figs. 3 (a) and (b) show the average DEP and total expected throughput vs. the node density as the number of slots per data collection cycle is varied. We observe that the average DEP using the Gaussian approximation closely matches the simulation results. Furthermore, from Fig. 4 we notice that the Gaussian approximation is a tight approximation to the exact analytic DEP value given by (30), even for low values of $K$.

We also notice that the general shape of the average DEP curve is the inverse image of the total expected throughput curve. This implies that as the throughput increases the DEP at the CH decreases. We would expect this because as more packets are received at the CH, more local decisions are available to improve the accuracy of the global decision. From the curves we see that the node density that minimizes the average DEP is $\lambda^* \approx 160$. This matches with the analytic expression $\lambda^* = \frac{1}{\pi R^2} \lambda_T^*, \xi$, which can easily be derived by solving $\frac{dS}{d\xi} = 0$, where $S$ is given by (36). The optimal node density also corresponds to the point at which the load on the channel is $G^* = 1$. This result is again what we would intuitively expect as a load smaller or greater than 1 would decrease the total expected throughput and hence increase the average DEP. Finally, we observe from Fig. 3(a) that as $\lambda$ increases the average DEP eventually converges to 0.5, since after $\lambda^*$ becomes very large the node density will cause the channel load to be much greater than one. Such large channel loads mean that there is a collision in almost every slot, causing the throughput to go to 0. Hence, when very few packets are

<table>
<thead>
<tr>
<th>System parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of cluster $R$</td>
<td>50 meters</td>
</tr>
<tr>
<td>Length of a slot $\ell$</td>
<td>1 second</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.05</td>
</tr>
<tr>
<td>Receiver sensitivity $\tau_{sen}$</td>
<td>-90dBm</td>
</tr>
<tr>
<td>Transmit power $P_{tx}$</td>
<td>-10dBm</td>
</tr>
<tr>
<td>Operating frequency $f$</td>
<td>2.4GHz</td>
</tr>
<tr>
<td>Reference distance $d_0$</td>
<td>1 meter</td>
</tr>
<tr>
<td>Path Loss at $d_0$ $P_{Lo}$</td>
<td>$\approx 40$dB</td>
</tr>
<tr>
<td>dB-spread of shadowing $\sigma_x$</td>
<td>4dB</td>
</tr>
<tr>
<td>Path loss exponent $\alpha$</td>
<td>2</td>
</tr>
<tr>
<td>Sampling time period $T$</td>
<td>0.5 seconds</td>
</tr>
<tr>
<td>Target parameters :</td>
<td></td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>1</td>
</tr>
<tr>
<td>$E_{tx}$</td>
<td>6500J</td>
</tr>
<tr>
<td>Energy parameters :</td>
<td></td>
</tr>
<tr>
<td>(per packet)</td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>42nJ/m^2</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.21mJ</td>
</tr>
<tr>
<td>Initial energy per mote $E_0$</td>
<td>100mJ</td>
</tr>
</tbody>
</table>
Fig. 3. (a) Comparison of the Gaussian approximation and simulation results for the average DEP vs. node density for $K = [20, 50, 100]$. In all cases the approximate analytic solution is essentially equal to the simulated values. (b) The total expected throughput vs. node density for different values of $K$. Note that the node density that maximizes the total expected throughput also minimizes the average DEP at the CH.

Fig. 4. A comparison of the Gaussian approximation with the exact analytic expression for the average DEP given by (30). We conclude that the Gaussian approximation is a tight approximation to the exact DEP value even for quite low values of $K$.

reached at the CH, the best global decision is to choose either of the hypothesis with probability 0.5.

Next we perform numerical experiments to study the network optimization problem formulated earlier. We first consider the case in which we impose an upper bound on the total number of sensor nodes that can be deployed. We use the following parameters for this scenario: we set $K = 100$; $\delta$ is varied from 0.1 to 1; and several values for the maximum number of nodes are considered: $N_{max} = 100, 200, 500, 1000, 1500$. The rest of the parameter values are the same as before.

Figs. 5 (a) and (b) show the maximum expected throughput per slot and the minimum average DEP that can be achieved using the optimal values of $N^*_T$ and $\Lambda^*_p$ for different values of $N_{max}$. We observe that the maximum throughput per slot converges to a value of about 0.36, which occurs when the channel load is 1. This value agrees with the theoretical value for the maximum throughput for the slotted ALOHA MAC.

Fig. 6 (a) shows the corresponding optimal node density value that is required to obtain the minimum DEP shown in Fig. 5 (b). For example, if one were to have the constraint that the maximum number of sensor nodes available is 1500, then using any value for $\lambda^*$ corresponding to the curve for $N_{max} = 1500$ in Fig. 6 (a) and depending on the user’s requirement in terms of the fraction of $E_0$ that can be used in one data collection cycle, the minimum DEP that can be achieved is given by the curve in Fig. 5 (b) corresponding to $N_{max} = 1500$.

From Figs. 5 and 6 we notice that there is an interesting trade-off between the energy used per data collection cycle, the minimum DEP that can be achieved and the optimal node density/total number of nodes required. Hence a particular minimum DEP value can be achieved by either having a large value for $N_{max}$ and using a small value of $\delta$, or a large value of $\delta$ and a small value for $N_{max}$. In the second scenario, we consider the case in which there are no constraints on the total number of sensor nodes that can be deployed in the field. For this situation, we use the values $K = 20, 50, 100$ while keeping rest of the parameters the same as before. Fig. 6(b) show the optimal node density value required to obtain the minimum average DEP for a particular value of $K$. Notice that in this case the minimum average DEP achievable is constant for each curve. Hence the trade-off here is that for a particular value of $K$ one can achieve the minimum average DEP using a higher value of $\delta$ with a small node density or a lower value of $\delta$ with a large node density. We also observe that increasing the value of $K$ would result in the CH receiving more decisions/packets from the sensor nodes and hence would decrease the minimum average DEP achievable.

VI. CONCLUSIONS

We studied the problem of determining the optimal spatial node densities and mean wake-up scheduling rate for deployments of a WSN for a target detection application. Unlike previous approaches, our objective was to find the parameters values that would minimize the overall average DEP at the CH, subject to practical system constraints such as the lifetime of the WSN and designer imposed cost/budget limitations. Our problem formulation accounted for all factors that affect the
The maximum expected throughput per slot vs $\delta$ achievable with different values of $N_{\text{max}}$, all with $K = 100$. Notice that the maximum value converges to about 0.36 as expected since this is the maximum theoretical expected throughput per slot for slotted ALOHA. (b) The minimum average DEP that can be achieved at the CH vs $\delta$ for different values of $N_{\text{max}}$ and using the optimal parameter values for $N^*_T$ and $\Lambda^*_p$. Similar to the throughput plot, the DEP eventually converges to a constant value.

design of WSN, including the underlying MAC protocol used, defective sensor motes, and signal-propagation characteristics. The results showed interesting tradeoffs between the number of sensor nodes deployed, the lifetime of the WSN, the minimum achievable overall average DEP at the CH, and the time/delay to reach a decision. The general optimization framework developed provides a designer the tools necessary to fine tune the network parameters to meet their specific requirements.

REFERENCES


