State Propagation for Targets Moving in Ground Terrain

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Abstract – This paper addresses the problem of propagating the state for a target moving in terrain, as a part of a tracking problem. The propagation is based on the assumption that the target has a goal, unknown to the observer, and that it chooses the fastest or otherwise least costly path to the goal. This is handled as a variational calculus problem, and the corresponding Euler-Lagrange equations thus lead to the state propagation equations. It turns out that the motion is very similar to that of light in media with varying index of refraction. The applicability of the method is discussed with particle filtering in mind.

Keywords: Ground target, tracking, terrain, propagation, variational calculus, Euler, Euler-Lagrange, refraction, particle filtering

1 Introduction

In target tracking one of the steps consists in propagating the knowledge about target state from one point of time to a later. This is made with the help of some assumption about target dynamics. Often very simple models are used, such as propagating with constant velocity. In ground target tracking, the question arises what principles to invoke when inhomogeneous terrain lies ahead of the target. This problem can be quite important as the prediction times are often quite long due to scarcity of sensors.

Most of the work in this field relates to road constrained targets, and understandably so as most of the ground targets prefer moving on roads. Less work has been done for off-road motion, or when on-road motion is combined with off-road. A special case of particular interest to us is in connection with particle filtering. A cloud of particles is propagated to a later time, and the particles can land in different terrain types. But how should we incorporate the terrain information? By adjusting particle weights according to terrain? By modifying the propagation path? What do we do when particles hit a hard constraint?

This has been studied by Ristic, Arulampalam and Gordon in [1], where the solutions steered towards different modes of motion, and their transitions in between. Another work in similar spirit is Ekman and Sviestins [2]. An interesting approach has been described by Fosbury et.al. [3], where the state vector is deflected towards terrain with high trafficability. [3] also gives a good overview over the ground tracking area, which will not be repeated here. Other more recent relevant work can be found in Roberts, Marshall and Powell [4] where classification is incorporated, and Christou [5], focusing on crowds of people.

A basic assumption in this paper is that the target has a goal, and that it tries to take the optimal, i.e. the fastest or least costly, path to the goal. It is further assumed that the speed is determined by the terrain type. There are techniques to determine this path, see e.g. [6] or [6]. Here, however, we take a different approach. Following variational calculus, a necessary condition for the least expensive path is that it satisfies the Euler-Lagrange equations. In other words, the state space model

\[
\frac{ds}{dt} = f(x) + G w_a(t) \tag{1}
\]

(1)

(where \(w_a(t)\) is a noise vector and \(G\) a mapping) should reflect the Euler-Lagrange equation. Equation (1) can then be used to propagate the current state estimate, regardless of tracking technique, e.g. particle filtering or unscented Kalman filtering.

Figure 1 The optimal path from A to B can be a combination of field, forest and road.

It should be noted that the cost or time optimization may lead to results that sometimes look counter-intuitive. If, for
example we are on a field, and our goal is on the other side of a forest, the fastest path is in general not the shortest, nor the one that avoids the forest entirely, but something in between as Figure 1 illustrates. It appears as though the path is deflected into the forest.

The techniques in [6] and [6], as well as in this paper, assume that the target speed depends on terrain type and vehicle type, according to a predictable relation. This is obviously an approximation. A couple of examples are [8] and [9].

In this paper we will present the underlying hypothesis in more detail, derive the Euler-Lagrange equations of motion, and discuss the consequences for target tracking.

2 Equations of Motion

2.1 Assumptions

The equations that are to be developed are based on some assumptions. These are more or less realistic, and when implementing the technique one must consider cases where the assumptions do not hold.

Assumption 1: The target is at a point \( A \) and has a goal \( B \). The goal is not known to us.

Assumption 2: The target takes the optimal, i.e., least costly path from point \( A \) to point \( B \).

Assumption 3: The cost can be described in terms of a cost function \( q(r) \) where \( r \) is the target position, and where the cost for dwell time \( dt \) at \( r \) is \( q(r)dt \).

Comment: With \( q = 1 \) the total cost is simply the time for getting from \( A \) to \( B \).

Assumption 4: It is possible to characterize the terrain by a function \( n(r) \) that determines the speed through

\[
v(r) = \frac{n_0}{n(r)} v_0 + \frac{C_1}{n(r)}
\]

where the speed \( v \) is given by

\[
v(r) = |v(r)| = \frac{|dr|}{dt}
\]

\( n_0 = n(r(0)) \), \( v_0 = v(r(0)) \) and \( C_1 \) is a constant. From the analogy with optics we call \( n(r) \) the index of refraction.

Comment 1: \( n \) obviously depends on the target type and terrain, as indicated in the introduction. But it can also depend on target speed. For example, if a fast target on a field drives into a forest, the speed is reduced drastically, while if the same target drives very slowly on the field it may keep almost the same speed in the forest. Going uphill may perhaps also be modeled by an increased \( n \).

Comment 2: Deviations from (2) will be handled as process noise in the final dynamic equations.

Assumption 5: The target and the observer share a common understanding about the terrain and the cost.

2.2 Problem formulation

Find the dynamic equations governing the target while moving from \( A \) to \( B \) in the least costly way, i.e., in the way that the trajectory \( \{r(s), t(s)\} \), \( s \in [0,1] \) minimizes the cost functional

\[
J[r(s)] = \int_0^1 q(r(s)) \frac{dr}{ds} ds
\]

where \( s \) is a parameter, the endpoints \( r(0) \) and \( r(1) \) correspond to point \( A \) and point \( B \), and \( t(s) \) is the corresponding time along the trajectory.

2.3 Solution

Using (2) and (3) the cost functional can be rewritten as

\[
J[r(s)] = \int_0^1 q(r(s)) \frac{dr}{ds} ds
\]

\[
= \int_0^1 \left( \frac{1}{v(r(s))} q(r(s)) \right) \frac{dr}{ds} ds
\]

\[
= \int_0^1 \left[ n(r(s)) q(r(s)) \right] \frac{dr}{ds} ds
\]

This is a classical variational problem, where one looks for a function \( y(s) \) minimizing or maximizing a functional

\[
\mathcal{L}[y(s)] = \int_a^b L(y(s), y'(s), s) ds
\]

A solution must satisfy the Euler-Lagrange equation

\[
\frac{\partial L}{\partial y} - \frac{d}{ds} \frac{\partial L}{\partial y'} = 0,
\]

see e.g. [10]. In this case we have two dependent variables \( r \) instead of the single \( y \). Therefore there will be two equations that we can jointly express as

\[
\frac{\partial}{\partial r} \left( n(r) q(r) \frac{dr}{ds} \right) = \frac{d}{ds} \frac{\partial}{\partial r} \left( n(r) q(r) \frac{dr}{ds} \right)
\]

Introducing
\[ g = \frac{d}{dr} \log n \]  
(9)

and

\[ h = \frac{d}{dr} \log q \]  
(10)

and remembering that

\[ \frac{d}{da} |a| = \frac{a}{|a|} \]  
(11)

for any vector \( a \), the Euler-Lagrange equation (8) runs

\[ nq(g + h) \frac{d}{ds} \frac{dr}{ds} = \frac{d}{ds} \left( nq \frac{dr}{ds} \right) \]

\[ = nq \left( (g + h) \frac{dr}{ds} \right) \frac{dr}{ds} + nq \frac{d}{ds} \left( \frac{dr}{ds} \right) \]

(12)

Note that

\[ \frac{d}{ds} = \frac{dt}{ds} \frac{d}{dt} \]

(13)

and

\[ \frac{dr}{ds} = \frac{dt}{ds} \frac{dr}{dt} = \frac{dt}{ds} \frac{v}{dt} \]

(14)

Thus \( \frac{dt}{ds} \) and \( nq \) are cancel out from the equation and we have simply

\[ \frac{d}{dt} \frac{v}{v} = (g + h)v - ((g + h) \cdot v) \frac{v}{v} \]

(15)

Equation (15) describes the rate of change of the unit vector along the trajectory, i.e. the turn rate times a unit vector across the trajectory.

To describe the propagation completely we need not only the direction, but also the speed. This is given by equation (2) which after differentiating and using (9) shows that

\[ \frac{dv}{dt} = - v \cdot g \]  
(16)

Combining equation (15) and (16) we finally have

\[ \frac{dv}{dt} = \frac{dv}{dt} v + v \frac{d}{dt} \left( \frac{v}{v} \right) \]

\[ = v^2 (g + h) - ((2g + h) \cdot v) v \]

(17)

2.4 Interpretation

![Diagram](image)

Figure 2 High \( n \) areas act repulsively to targets moving straight into it, while attracting those moving across the gradient.

In (17) \( g \) and \( h \) enter in almost the same way; to understand the difference, suppose for a moment that \( h = 0 \). We can see that when moving right into an area with high \( n \), i.e. \( v \) parallel to \( g \), we have

\[ \frac{dv}{dt} = -v^2 g \]

(18)

which says that the area acts repulsively and the target slows down, as expected, see Figure 2. On the other hand, when \( v \) is perpendicular to \( g \), we have

\[ \frac{dv}{dt} = v^2 g \]

(19)

so the high \( n \) area “pulls” the target into it. This is in accordance with the introduction and also with how refraction works in optics. On the other hand, in the case that \( g = 0 \), the speed is not affected when going right into the high \( q \) area. In the perpendicular direction \( h \) acts the same way as \( g \).

2.5 Special case: Small Gradients

There is of course no general explicit solution to (1), however, we may point out that if \( g \) and \( h \) are small then the first order perturbation to \( v \) is governed by a much simpler system of equations.
\[ \mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} \]
\[ \mathbf{v}^{(0)} = \text{const.} \quad (20) \]
\[ \frac{d\mathbf{v}^{(1)}}{dt} = \mathbf{v}^{(0)2}(g + h) - \left[(2g + h) \cdot \mathbf{v}^{(0)}\right] \mathbf{v}^{(0)} \]

### 2.6 Special case: Constant Gradient Direction, and Snell’s Law

An important special case is when \( n \) and \( q \) depend on only one spatial coordinate, i.e., there is a coordinate system \((x, y)\) where \( g_x = g(y), g_y = 0 \) and \( h_x = h(x), h_y = 0 \). A typical example could be when travelling in a region with constant \( n \) and \( q \) except for a boundary where the values change.

\[
\begin{align*}
\frac{dv_x}{dt} &= v^2(g + h) - 2gv_x^2 - hv_x^2 \\
\frac{dv_y}{dt} &= -2gv_xv_y - hv_xv_y
\end{align*} \quad (21)
\]

Now consider the quantity

\[ C_2 = v_y n^2 q \quad (22) \]

Differentiate \( C_2 \) with respect to \( t \), recall (9) and (10) and compare with (21). This shows that \( C_2 \) is a constant of motion. Moreover, use (2) and it follows that

\[ C_3 = \frac{C_2}{C_1} = \frac{v_y}{v} \quad (23) \]

too is a constant of motion. Here we have rediscovered Snell’s law which describes how light rays refract when passing from one medium to another. Snell’s law is usually expressed as

\[ n_1 q_1 \sin \theta_1 = n_2 q_2 \sin \theta_2 \quad (24) \]

where the \( \theta_1, \theta_2 \) are the angles between the rays and the forward directed normal to the surface, before and after the passage, see Figure 3, but here we also include the cost function \( q \).

![Figure 3. Snell’s law of refraction](image)

In reality using Snell’s law will probably provide the most useful way of propagating the state vector, however there is another that may be worth writing down.

Eliminate \( v_y \) from (2) and (22). This gives

\[ v_x^2 = \frac{C_1^2 - C_2^2}{n^2 - n'^2 q^2} \quad (25) \]

Then we can write

\[ dt = \pm \sqrt{\frac{dx}{\frac{C_1^2 - C_2^2}{n^2 - n'^2 q^2}}} \quad (26) \]

and thus \( t \) as a function of \( x \). For some easy shapes of \( n(x) \) and \( q(x) \), this can be integrated explicitly and even inverted to provide \( x(t) \), and then via (22), at least implicitly, \( y(t) \). However the results are complicated and are probably of very limited practical value.

### 2.7 State propagation model

Equation (1) describes the propagation in general of a state space vector. From (17) we can immediately write equation (1) for our specific case:

\[ \frac{d}{dt} \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix} = \left( v^2(g + h) - \left[(2g + h) \cdot \mathbf{v}\right] \mathbf{v}\right) + \begin{pmatrix} 0 \\ \mathbf{w} \end{pmatrix} \quad (27) \]

It is in general not possible to integrate this to find explicit discrete time equations, however, the following three methods can be considered:

(a) Assume that \( n \) and \( q \) are constant (i.e. \( h = g = 0 \) ) except at the boundaries. Apply elementary kinematic relations \( (\mathbf{v} = \mathbf{v}_0, \mathbf{r} = \mathbf{r}_0 + \mathbf{v} t) \) outside the boundaries. When passing the boundary, change the speed according to the index of refraction \( n \) and change direction according to Snell’s law.

(b) Approximate the propagation model by assuming \( g \) and \( h \) small, and see if (20) can be solved exactly.
(c) Apply numerical integration of (27) e.g. by dividing the propagation time into small steps.

In practical applications one can expect that method (a) will be the most useful, not least because terrain data is usually published in the form of areas with certain properties, e.g. a polygon representing a field.

3 Tracking issues

So far the description has been focused on the target and its behavior. We can now change perspective to that of the observer, i.e., the tracker. If Assumption 5 (Section 2.1) holds true, then it should be possible to use the observer’s knowledge about the target to predict the path.

By formulating the state propagation model (27) we have turned the boundary value problem (4) into an initial value problem. Although the goal is not known to the observer, by propagating from the current position and velocity, the goal should be somewhere along the predicted path. In reality however the initial state is not exactly known; it may be represented e.g. by a state vector and a covariance, or by a particle cloud. This will be illustrated with three examples on the propagation of a particle cloud, where each of the particles is propagated using Snell’s law.

Figure 4. Moving into dense area \( n = 4 \). A particle cloud is shown at times 0, 10, 20, 30 and 40 seconds.

In Figure 4 a target starts at position \((-90,0)\), velocity \((5,5)\) and \( n = 1 \) and enters a dense area, perhaps a forest, covering the half-plane with \( x > 0 \). One can clearly see how the direction changes at the border, and also how speed is reduced.

Figure 5. Moving from a dense region to a less dense region at 0, 10, 20 and 30 seconds.

Next, Figure 5 shows the opposite case where the target moves from a high density region into a less dense \( n = 1 \) and \( n = 0.7 \) respectively. Now the cloud is deflected from the normal, its speed increases and the particles spread.

Figure 6. Moving into a hard constraint

Figure 6 shows what happens when moving into a hard constraint, e.g. a lake. The particles stop there. This is
actually completely in accordance with the initial assumptions. If the goal of the target was somewhere else, it would not head right into the lake. So the destination must be either the lake, or some point on the way to the lake.

This way of propagating system state and state estimates is attractive in that it is clearly model based. However, when the model fails to describe reality one may need to expand the model, or introduce suitable workarounds. Here we indicate some cases when other methods may have to be considered.

Figure 7. The goal is on the other side of a hard constraint, e.g. a lake. Only one of the prediction lines leads behind the lake.

Problems arise when touching the border of a hard constraint, see Figure 7. The target’s goal is on the other side of the lake, and the optimal path obviously has to follow the shore for some distance. But predictions starting with just slightly different directions go wrong. Either they end at the shore, or they do not bend. Moreover, for the line that actually touches the lake, it is not possible to predict at which point it will depart from the shore.

Figure 8 displays a similar case with a target on road. The optimal path follows the inside of the bend. Prediction lines on either side of this direction end prematurely.

The problems in Figure 7 and Figure 8 are related to certain geometries in combination with very high values of the gradients \( g \) and \( h \). The conclusion that can be drawn is that standard particle filtering will not work well here, and other techniques have to be considered.

Another peculiarity that should be observed is the following. When going from a high \( n_q \) to a low \( n_q \) it may easily happen that the incidence angle \( \theta_1 \) (Figure 3) is so large that there is no \( \theta_2 \) satisfying Snell’s law. For example, if \( n_q \) is reduced by a factor 4, then the maximum incidence angle is only 14.5 degrees. In optics this effect leads to total reflection. Here it means either that the target does not follow the optimal path, or that the goal is on the target’s side of the border. It is not obvious how to handle such cases.

Finally we have the general problem of how to handle cases when the target does not follow the optimal path for whatever reasons: for example target, observer or both have incomplete knowledge about \( n_q \), or the target may be patrolling, exercising or improvising and therefore not have a goal.

4 Conclusion

Using calculus of variations, and the model assumption that a target takes the least costly path to reach its goal, we have derived surprisingly simple equations of motion. The results show that the corresponding behavior resembles light in media with varying index of refraction.

But all models of reality have their limitations, and what we present here is not the final solution to the terrain tracking problem. Better modeling will be needed, e.g. by making some kind of synthesis with \[3\]. Improved filtering techniques should also be developed so that particles are not wasted as in the cases with steep boundaries. We are keen on exploring these areas, and we do believe the presented approach is a step towards good ground tracking with sparse measurements.

References


