Abstract—The state of some practical dynamic systems satisfies constraints, which can be utilized to improve the performance of state estimation. State estimation with nonlinear inequality constraints is a challenging problem. Projection methods are widely used to solve this problem. In this paper, a projection method is formulated as a special nonlinear function. Based on this formulation, unconstrained estimated states can be easily projected into the feasible constraint region through unscented transformation (UT), and both the constrained mean and covariance can be obtained. An approach is proposed to reduce the complexity of this projection process and a closed-form solution is derived for one-dimensional constraints. To solve the problem that the constrained covariance may be ill-conditioned in UT-based approaches, a modification of the covariance is also proposed to account for necessary uncertainties. Two illustrative scenarios for ground moving target tracking are simulated to demonstrate the effectiveness and the efficiency of the proposed approaches.

Keywords: State estimation, nonlinear inequality constraints, unscented transformation, target tracking.

I. INTRODUCTION

In practice, the state of some dynamic systems satisfies some constraints due to several reasons [1], e.g., basic laws of physics, mathematical descriptions and practical limitations. Such constraints are generally described as a combination of linear or nonlinear equalities or inequalities. Properly utilizing the information provided by the constraints can improve the accuracy of state estimation [2]. However, incorporating the constraints into the estimation process is challenging, especially when the constraints are expressed as nonlinear inequalities.

Many approaches have been proposed for state estimation with constraints. They include the pseudo-observation, projection, reparameterization methods and others (see, e.g., [1, 3–5]). The reparameterization methods (see, e.g., [2, 6]) attempt to reparameterize the system so that the equality constraint is naturally satisfied. These methods are not easy to apply to the system with nonlinear inequality constraints, and are also beyond the scope of this paper. In this paper, only the first two methods are discussed for real-time applications.

In the pseudo-observation method (e.g., [7, 8]), the equality constraints are treated as perfect observations (without noise) and then a typical filter, e.g., Kalman filter (KF), is applied to obtain the estimation results. Although this method has several merits, e.g., optimality and clear intuitive interpretations, it may suffer from numerical problems caused by a singular covariance matrix of measurement error and an increased computational burden caused by augmenting measurements [9]. Further, it is difficult to apply this approach to the system with nonlinear inequality constraints.

The projection method (e.g., [1, 9–11]) projects the estimation results onto the constraint surface or into the feasible region. This method can be applied to systems with equality or inequality constraints. For state estimation with nonlinear equality constraints, [1] identified two types of constraints for the estimated states: Type I (those that act on the entire distribution) and Type II (those that act on the mean of the distribution), and further pointed out that the pseudo-observation method enforces neither type of constraints and that the projection method (adopted in that paper) enforces the first type of constraint only. A two-stage approach was then proposed in [1] to ensure that the second type of constraint is also satisfied in the projection method. For a system with linear inequality constraints, [12] projected the unconstrained state estimate (conditional mean) onto some constrained set by solving a convex quadratic programming problem. This approach can also be applied to nonlinear inequality constraints by linearizing the nonlinear function around the unconstrained conditional mean. The moving horizon based estimators (MHE) [3] [4] have been proposed to naturally ensure that the estimates satisfy constraints by solving a constrained optimization problem. In a non-recursive form, this approach is computationally inefficient and may be difficult to use for real-time applications [15].

On the other hand, constraints also affect the conditional error covariance of the estimated state. This fact was considered in [1] and the constrained covariance was calculated as in the extended KF (EKF) when the projection function is differentiable. For the case that the state is constrained with hard bounds, [13] also proposed an approach to calculate the constrained mean and covariance based on the unscented transformation (UT) [14] where the sigma points that violate the inequality are projected normally onto the boundary of feasible region. Although the mathematical form of this projection
method for every sigma point is not given in [13] (illustrated by several figures instead), the adopted projection method is actually a special case of the one in [12]. Approaches with a similar idea can also be found in [15] and [16]. In these papers, it was shown that incorporating the constraint information into both the estimated mean and covariance can improve estimation performance. Because obtaining the distribution of the constrained state conditioned on measurements is very difficult, using UT in the projection methods to calculate the constrained mean and covariance seems attractive, as adopted by the above approaches. However, this approach faces two problems:

(a) Projecting every sigma point into the feasible constraint region may be computationally inefficient, especially for high dimensional state vectors and for the case where the projected point is derived by solving an optimization problem.

(b) Directly using the projected sigma points to calculate the covariance may make it ill-conditioned, especially for nonlinear inequality constraints. This phenomenon will be illustrated by a simple example later.

This paper attempts to solve these two problems. First, a general formulation is given to clearly show the projection process based on optimization methods for nonlinear inequality constraints. Based on this formulation, UT is adopted to calculate the constrained mean and covariance. An approach is also proposed to relieve the computational burden, and a closed-form solution for one-dimensional constraints is also derived. For the second problem, a modification of the constrained covariance is proposed, and the connections between this method and some other approaches are also discussed.

This paper is organized as follows. Section II presents the above approaches as the main contents of this paper. Section III investigates an illustrative scenario for ground moving target tracking and presents simulation results and discussions. Section IV provides conclusions.

II. STATE ESTIMATION WITH NONLINEAR INEQUALITY CONSTRAINTS BASED ON UT

The following model with constraints is considered:

\[
\begin{align*}
    x_k &= F_k x_{k-1} + w_{k-1} \\
    z_k &= H_k x_k + v_k \\
    L_k &\leq c_k(x_k) \leq U_k
\end{align*}
\]

where \( k \) is the time index, \( x \) the state vector, \( z \) the noisy measurement vector, \( w \) the process noise, \( v \) the measurement noise, \( F \) the transition matrix of the state \( x \), and \( H \) the measurement matrix. \( w_{k-1} \) and \( v_k \) are assumed to be mutually uncorrelated zero-mean Gaussian white sequences with covariance matrices \( Q_{k-1} \) and \( R_k \) respectively. \( c_k(x_k) \) is the constraint function and \( U_k \) and \( L_k \) are the upper and the lower bounds of \( c_k(x_k) \) respectively.

The purpose is to estimate \( x_k \) given measurement sequence \( Z^k \triangleq \{z_1, ..., z_k\} \) based on model (1). Here the projection method is considered.

Assume that the unconstrained estimation results are given, i.e., the unconstrained state estimate \( \hat{x}_{k|k} \) and the error covariance matrix \( P_{k|k} \) are provided (by using the Kalman filter or Kalman-type filters). One typical projection method can be given as (Type I constraint)

\[
\hat{x}^P_{k|k} = \arg\min_y (y - \hat{x}_{k|k})^T W_k (y - \hat{x}_{k|k})
\]

s.t. \( L_k \leq c_k(y) \leq U_k \)

where \( W_k \) is any symmetric positive definite weighting matrix. Note that the projection can also be applied to the one-step prediction \( \hat{x}_{k|k-1} \). If \( c_k(y) \) is linear in \( y \) and \( L_k = U_k \), (2) reduces to the projection approach proposed in [9]. If \( c_k(y) \) is linear with \( L_k \neq U_k \), (2) is actually the same as the method proposed in [12]. In [9] and [12], it was proved that among all the possible filters with the projection method (2) for a linear \( c_k(y) \), the filter that uses \( W_k = P_{k|k}^{-1} \) has the smallest estimation error covariance. It was also pointed out that if \( c_k(y) \) is nonlinear, those approaches can also be applied by linearizing \( c_k(y) \) around \( \hat{x}_{k|k} \) as in the EKF.

In fact, (2) projects only the unconstrained mean \( \hat{x}_{k|k} \) rather than the whole distribution into the feasible region. [13] proposed an approach to calculate both the constrained mean \( \hat{x}^P_{k|k} \) and error covariance \( P^P_{k|k} \) based on UT. In this approach, the sigma points are generated from the unconstrained mean and covariance first, and the points outside the feasible region are projected onto its boundaries. The final results are calculated based on the projected sigma points. However, the mathematical expression of the projection process is not given. To analyze this approach, we propose a general formulation of the projection process based on the optimization method (2). Assume that the unconstrained \( \hat{x}_{k|k} \) and \( P_{k|k} \) are the mean and the covariance matrix of random vector \( x_{k|k} \) respectively:

\[
\begin{align*}
    \hat{x}_{k|k} &= E\{x_{k|k}\} \\
    P_{k|k} &= E\{(x_{k|k} - \hat{x}_{k|k})(\cdot)^T\}
\end{align*}
\]

where \( (\cdot) \) means the item before. In fact, \( x_{k|k} \) represents any random vectors with the first two moments equal to \( \hat{x}_{k|k} \) and \( P_{k|k} \) respectively, although these random vectors may have different distributions.

Different from (2), we consider projecting all the possible realizations of \( x_{k|k} \) into the feasible region rather than just its mean \( \hat{x}_{k|k} \). This projection process can be described as

\[
\begin{align*}
    x^P_{k|k} &= \arg\min_y (y - x_{k|k})^T W_k (y - x_{k|k}) \\
    \text{s.t. } L_k &\leq c_k(y) \leq U_k
\end{align*}
\]

Then the constrained mean and error covariance can be calculated as

\[
\begin{align*}
    \hat{x}^P_{k|k} &= E\{x^P_{k|k}\} \\
    P^P_{k|k} &= E\{(x^P_{k|k} - \hat{x}^P_{k|k})(\cdot)^T\}
\end{align*}
\]

The approach in (4) can actually project the distribution of the unconstrained state into the feasible region. However, directly applying (4) to obtain \( \hat{x}^P_{k|k} \) and \( P^P_{k|k} \) is difficult, since the specific distribution of the unconstrained state is hardly known.
and based on UT faces two problems: to project each sigma point into the feasible region by using simulation section. As described above, this approach needs application of this approach will be demonstrated in the sequel. By this definition, random vector \(x_k\) will be on the boundary of the feasible region, which makes why the UT was applied for projection in [13] [16]. However, as stated in Section I, this projection approach even if its first two moments are given. To solve this problem, an approach based on UT is presented next.

A. Projection Approach Based on UT

Since only the first two moments \(\hat{x}_{k|k}\) and \(P_{k|k}\) of \(x_{k|k}\) are given, UT is considered to solve the problem above. For this purpose, the following nonlinear function is defined to embody the projection process (4):

\[
x_{k|k}^P \triangleq p(x_{k|k}) = \operatorname{arg} \min_y (y - x_{k|k})'W_k(y - x_{k|k})
\]

s.t. \(L_k \leq c_k(y) \leq U_k\) \hspace{1cm} (6)

Function \(p(x_{k|k})\) will be referred to as the projection function in the sequel. By this definition, random vector \(x_{k|k}^P\) is just a nonlinear function \(p(x_{k|k})\) of another random vector \(x_{k|k}\) with known first two moments. This may be the main reason why the UT was applied for projection in [13] [16]. However, \(p(x_{k|k})\) is a special nonlinear function that may be non-differentiable at some points and most realizations of \(x_{k|k}^P\) will be on the boundary of the feasible region, which makes directly applying UT here for the final results questionable. This will be analyzed later. Simply speaking, this UT-based projection process can be described as

1) Generate sigma points \(\{s_k^i\}_{i=1}^{2n+1}\) based on \(\hat{x}_{k|k}\) and \(P_{k|k}\).

Here \(n\) is the dimension of \(\hat{x}_{k|k}\).

2) Calculate \(s_k^{P,i} = p(s_k^i)\) for \(i = 1, ..., 2n + 1\) as in (6).

3) Calculate \(\hat{x}_{k|k}^P\) and \(P_{k|k}^P\) based on \(\{s_k^{P,i}\}_{i=1}^{2n+1}\) as in the unscented filter [14].

Actually, the projection function of [13] (shown by several figures) can be viewed as a reduced form of (6) with \(c_k(y) = y\) and \(W_k = I\). The reason is that in [13], the sigma points outside the feasible region constrained by the upper and the lower bounds are directly assigned with the corresponding bound values (with other points unchanged), which is actually the solution of (6) for each sigma point with \(c_k(y) = y\) and \(W_k = I\) (see [17] for more details). Approaches with similar ideas can also be found in [16].

However, as stated in Section I, this projection approach based on UT faces two problems:

(a) Projecting every sigma point into the feasible region by solving (6) may be computationally inefficient.

(b) Directly using the projected sigma points to calculate the covariance for this special projection function may be ill-conditioned.

The following two subsections attempt to solve these two problems respectively.

B. Simplification of the Projection Approach

In this subsection, an approach is proposed to reduce the computational complexity for the UT-based projection method. Application of this approach will be demonstrated in the simulation section. As described above, this approach needs to project each sigma point into the feasible region by using (6). By representing a sigma point (vector) as \(s_k\), (6) can be rewritten as

\[
s_k^P = \operatorname{arg} \min_y (y - s_k)'W_k(y - s_k)
\]

s.t. \(L_k \leq c_k(y) \leq U_k\) \hspace{1cm} (7)

It is well known that if the constraints are linear equalities, this optimization problem can be solved easily. That is, for the following problem,

\[
s_k = \operatorname{arg} \min_y (y - s_k)'W_k(y - s_k)
\]

s.t. \(C_k y = d_k\) \hspace{1cm} (8)

the solution is \([2, 17]\)

\[
s_k = s_k + M_k(d_k - C_k s_k)
\]

where \(M_k = W_k^{-1}C_k^T(C_k W_k^{-1}C_k^T)^{-1}\).

Based on the above results, we consider solving the following optimization problem with linear inequality constraints:

\[
s_k^P = \operatorname{arg} \min_y (y - s_k)'W_k(y - s_k)
\]

s.t. \(L_k \leq c_k(y) \leq U_k\) \hspace{1cm} (10)

This can be decomposed into two steps:

1) Solve (8) for \(s_k^P(d_k)\) as a function of \(d_k\).

2) Then solve

\[
d_k^P = \operatorname{arg} \min_{d_k} (s_k^P(d_k) - s_k)'W_k(s_k^P(d_k) - s_k)
\]

s.t. \(L_k \leq d_k \leq U_k\) \hspace{1cm} (11)

And the final result is given as \(s_k^P(d_k^P)\). By solving (8) and substituting (9) into (11) yields

\[
d_k^P = \operatorname{arg} \min_{d_k} (d_k - C_k s_k)'M_k W_k M_k(d_k - C_k s_k)
\]

s.t. \(L_k \leq d_k \leq U_k\) \hspace{1cm} (12)

Then the final result becomes

\[
s_k^P(d_k^P) = s_k + M_k(d_k^P - C_k s_k)
\]

A proof of the optimality of \(s_k^P(d_k^P)\) as the solution of (10) is given next.

**Proof:** Without loss of generality, assume \(s_k^0\) be the optimal solution of (10), then \(d_k^0 = C_k s_k^0\) satisfies the constraints, i.e., \(L_k \leq d_k^0 \leq U_k\). For any possible vector \(s_k\) that satisfies \(C_k s_k = d_k^0\), only \(s_k^0 = s_k + M_k(d_k^0 - C_k s_k)\) has the minimum \((s_k(d_k) - s_k)'W_k(s_k(d_k) - s_k)\) as given by (9), which means that \(s_k^0 = s_k^0 = s_k + M_k(d_k^0 - C_k s_k)\), since \(s_k^0\) is optimal.

That is to say, the optimal solution of (10) with condition \(C_k y = d_k, L_k \leq d_k \leq U_k\) (equivalent to the condition \(L_k \leq C_k y \leq U_k\)) must have the following linear form:

\[
s_k(d_k) = s_k + M_k(d_k - C_k s_k), L_k \leq d_k \leq U_k
\]
or it is not optimal otherwise. Substitution of (14) into (10) yields (replace \( y \) with \( s_k(d_k) \)):

\[
\begin{align*}
    s_k^P &= \arg\min_y (y - s_k)^T W_k (y - s_k) \\
    \text{s.t. } L_k &\leq C_k y \leq U_k \\
\end{align*}
\]

\[
\begin{align*}
    s_k^P &= s_k(d_k^P), \\
    d_k^P &= \arg\min_{d_k} \{ (s_k + M_k(d_k - C_k s_k) - s_k)^T x W_k (s_k + M_k(d_k - C_k s_k) - s_k) \} \\
    \text{s.t. } L_k &\leq C_k(s_k + M_k(d_k - C_k s_k)) \leq U_k \\
\end{align*}
\]

\[
\begin{align*}
    s_k^P &= s_k(d_k^P), \\
    d_k^P &= \arg\min_{d_k} (M_k(d_k - C_k s_k))^T W_k (M_k(d_k - C_k s_k)) \\
    \text{s.t. } L_k &\leq d_k \leq U_k \\
\end{align*}
\]

\[
\begin{align*}
    s_k^P &= s_k(d_k^P), \\
    d_k^P &= \arg\min_{d_k} (d_k - C_k s_k)^T M_k^T W_k M_k (d_k - C_k s_k) \\
    \text{s.t. } L_k &\leq d_k \leq U_k \\
\end{align*}
\]

where “\( \Leftrightarrow \)” means “is equivalent to”, \( M_k \) is given in (9), and \( C_k(s_k + M_k(d_k - C_k s_k)) = d_k \). The last “\( \Leftrightarrow \)” in (15) indicates the optimality of \( s_k^P \) in (13) as the solution of (10). This completes the proof.

Note that solving (12) may be much easier than solving (10) directly since in most cases \( C_k \) is a “fat” matrix with more columns than rows.

If \( C_k y \) is one-dimensional, a closed-form solution of (10) can be easily obtained by solving (12) as (13) and

\[
\begin{align*}
    d_k^P &= \begin{cases} 
        U_k, & C_k s_k \geq U_k \\
        L_k, & C_k s_k \leq L_k \\
        C_k s_k, & \text{elsewhere.} 
    \end{cases} 
\end{align*}
\]

If \( c_k(y) \) is nonlinear in \( y \), one common projection method is to linearize it around \( s_k \) [9,12]:

\[
c_k(y) \approx c_k(s_k) + Dc_k(s_k)(y - s_k) \]

and then the optimization problem (7) can be solved approximately as in (12). In (17), \( Dc_k(s_k) \) is the gradient of function \( c_k(.) \) evaluated at \( s_k \).

**Remark 1:** The constraints in some practical systems are one-dimensional. For example,

(a) In ground moving target tracking, the road constraints can be expressed as one-dimensional inequalities.

(b) In extended (cluster or group) object tracking, the distance in position between two objects of interest within the cluster satisfies some constraint, which can be formulated as one-dimensional inequality.

Thus, the above solution of the projection problem with a one-dimensional inequality can be useful in practice, which will also be demonstrated in the simulation section.

**C. State Estimation with Inequality Constraints Based on UT**

In this subsection, we consider the second problem pointed out in Subsection A: directly using the projected sigma points to calculate the covariance may be ill-conditioned. The main reason is that the projection function \( p(x_{k|k}) \) defined in (6) is not a “nice” nonlinear function—e.g., it is not differentiable at some specific points. This problem is analyzed next and an approach for covariance modification in the UT-based projection is proposed to solve it.

Take the case with \( c_k(y) = y \) and \( W_k = I \) in (6) for illustration. Under this assumption, \( U_k \) and \( L_k \) are the upper and the lower bounds for \( y \) respectively. Based on UT, the sigma points outside the feasible region defined by these bounds are directly assigned to the corresponding bound values by solving (6), as done in [13]. Consider two cases about this projection:

(a) Some of the sigma points are within the feasible region.

(b) None of the sigma points are within the feasible region.

The projection approach based on UT for these two cases are shown in Figures 1 and 2 respectively. In these figures the state vector \( x_k \) is assumed two-dimensional and the constraints are the upper and the lower bounds for each entry since \( c_k(y) = y \). Sigma points \( \{ s_k^i \}_{i=1}^5 \) are generated from \( \hat{x}_{k|k} \) and \( P_{k|k} \). In Figure 1 (Case 1), the projection process is similar to that of [13]. In this case, only \( s_k^1, s_k^2, s_k^5 \) which violate the constraints are normally projected onto the boundary of the feasible region (since \( W_k = I \)) to become \( s_k^1, s_k^2, s_k^5 \) respectively. Then the final results are calculated as

\[
\begin{align*}
    \hat{x}_{k|k}^P &= \sum_{i=1}^{2n+1} w_i s_k^i P_{k|k}^P \\
    P_{k|k}^P &= \sum_{i=1}^{2n+1} w_i (s_k^i P_{k|k} - x_k^P) (\cdot)^T 
\end{align*}
\]

where \( n = 2 \), \( w_i \) is the weight assigned to \( s_k^i (i = 1, \ldots, 5) \) in UT, and \( s_k^i P_{k|k} = s_k^i \) for \( i = 1, \ldots, 5 \) in Figure 1. By this projection, both mean and covariance of the estimated state are modified by the constraints. That is actually why this approach is better than those projecting only \( x_{k|k} \) into the feasible region [13].

![Figure 1. The Projection Approach Based on UT (Case 1)](image)
However, directly applying this projection approach to Case 2 is questionable as shown in Figure 2. In this case, all sigma points violate the constraints and are projected onto the boundary of the feasible region (left edge of the rectangle). Thus, $P_{k|k}^P$ calculated based on (18) is ill-conditioned as

$$P_{k|k}^P = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix} \quad (19)$$

Figure 2. The Projection Approach Based on UT (Case 2)
- sigma point, o projected sigma point.
The rectangle represents the constraints.

Using this $P_{k|k}^P$ for further estimation may make the estimation process diverge. In fact, $x_{k|k}$ can be distributed all over the feasible region even if the sigma points are all outside the region as in Figure 2, that is, the projected sigma points can not represent the constrained distribution, especially when hard constraints exist. This may be because the projection function $p(\hat{x}_{k|k})$ defined in (6) is actually not differentiable due to the hard constraints. In view of this, we consider modifying the calculation of $P_{k|k}^P$. Here for $P_{k|k}^P$, the sigma points are modified to include more covariance information lost in the standard projection. The new sigma points are generated as (based on (13))

$$\hat{s}_k^i = s_k^i + \lambda M_k(d_k^i - C_k s_k^i) \quad (20)$$

where $0 \leq \lambda \leq 1$ is a parameter selected for including more covariance information and will be discussed later. Then $\hat{x}_{k|k}^P$ and $P_{k|k}^P$ can be calculated as

$$\hat{x}_{k|k}^P = \sum_{i=1}^{2n+1} w_i \hat{s}_k^i \quad (21)$$

$$P_{k|k}^P = \sum_{i=1}^{2n+1} w_i (\hat{s}_k^i - \hat{x}_{k|k}^P)(\cdot)^t$$

For “common” nonlinear functions (e.g., differentiable at any point), the final mean and covariance can be calculated as in (18) based on UT (treat $p(x)$ as the functions). Compared with (18), the modified (21) is based on following considerations:

(a) The constrained mean $\hat{x}_{k|k}^P$ calculated by (18) and (21) respectively are the same. The projected sigma points $s_k^i$ are in the feasible region. If the region is convex, as the convex combination (with weights $w_i > 0$ and $\sum_i w_i = 1$) of $s_k^i$, $\hat{x}_{k|k}^P$ also satisfies the constraints. In this case, no modification needs to be made to $\hat{x}_{k|k}^P$ without any further information about the distribution of the constrained state.

(b) The constrained covariance $P_{k|k}^P$ in (21) is different from that in (18). The modification here is just to include more covariance information. For the standard projection, the only modification to the original sigma point $s_k^3$ is the term $M_k(d_k^i - C_k s_k^i)$, as shown in (13), which is also the main reason why the constrained covariance matrix becomes ill-conditioned. Thus, this term is scaled down by $\lambda$ in (20) to include the lost covariance information.

(c) Given more information about the distribution of the unconstrained estimated state, more realizations of $x_{k|k}$ can be obtained, and the corresponding projected points can be calculated by (6). Based on these projected points, more accurate constrained estimation results can be obtained. However, this may not be feasible for some practical applications since it is difficult to obtain that distribution. In this case, using the above approach is optional to solve the problem of ill-conditioned covariance matrix due to the UT-based projection method.

Different values of $\lambda$ have different meanings, as stated next.

**Remark 2:**
1) $\lambda = 1$ makes $\hat{s}_k^i = s_k^i$ and thus $P_{k|k}^P$ reduces to the standard covariance matrix obtained by the standard projection approach as in (18). In this case, the projection function is treated as a “common” nonlinear function and UT is directly applied.
2) $\lambda = 0$ yields $\hat{s}_k^i = s_k^i$ and

$$\hat{x}_{k|k}^P = \sum_{i=1}^{2n+1} w_i \hat{s}_k^i \quad P_{k|k}^P = \sum_{i=1}^{2n+1} w_i (\hat{s}_k^i - \hat{x}_{k|k}^P)(\cdot)^t = \sum_{i=1}^{2n+1} w_i (s_k^i - \hat{x}_{k|k}^P)(\cdot)^t = \sum_{i=1}^{2n+1} w_i (s_k^i - \hat{x}_{k|k} + \hat{x}_{k|k} - \hat{x}_{k|k})(\cdot)^t = P_{k|k} + (\hat{x}_{k|k} - \hat{x}_{k|k})(\cdot)^t$$

This actually makes this method equivalent to the displOff method proposed in [1]. This method actually indicates:

*By moving the estimate to a different value, the filter ceases to propagate the conditional mean, and the covariance must increase from $P_{k|k}$ to $P_{k|k}^P$.* [1].

3) $0 < \lambda < 1$ makes the covariance modification in a case between the above two.

Generally, $\lambda$ needs to be designed properly for different specific cases with different projection functions to obtain satisfactory performance. This is also discussed in the following simulation section.

**III. Simulation Results**

To illustrate the effectiveness of the proposed approach, two scenarios for ground moving target tracking were designed as follows:
(a) Scenario 1 (S1). The target was on a linear road satisfying the following linear constraint:

\[
R_L \leq C_k x_k \leq R_U \quad x_k \triangleq \begin{bmatrix} p^x_k, v^x_k, p^y_k, v^y_k \end{bmatrix}^T
\]

where \( k \) is time index, \( p^x_k, v^x_k, p^y_k, v^y_k \) are the positions and velocities of the target in the \( x \) and \( y \) directions, and \( C_k \triangleq [1 \; 0 \; 1 \; 0] \) is the constraint matrix. The target performed the CV (nearly constant velocity) motion along the road and the trajectories were uniformly distributed within the road.

(b) Scenario 2 (S2). The target was on a circular road satisfying the following nonlinear constraint:

\[
R_L \leq c_k(x_k) \leq R_U \quad x_k \triangleq \begin{bmatrix} p^x_k, v^x_k, p^y_k, v^y_k, \omega_k \end{bmatrix}^T
\]

where \( \omega_k \) is the turn rate, the nonlinear constraint function is \( c_k(x_k) \triangleq \sqrt{(p^x_k - p^x_c)^2 + (p^y_k - p^y_c)^2} \), \( (p^x_c, p^y_c) \) is the coordinates of the center of the circular road, and other notations are the same as in (23). The target performed the CT (nearly constant turn rate) motion along the road and the trajectories were uniformly distributed within the road.

In (23) and (24), \( R_L \) and \( R_U \) are the upper and the lower bounds of the road width respectively. The measurements are noisy positions of the target at each sampling time.

The above two scenarios are designed to evaluate the proposed algorithm under linear and nonlinear constraints respectively. Specifically, the corresponding parameters and filters are designed as follows.

For S1, the CV model was adopted in the algorithms, given by (1) with (18)

\[
x_k \triangleq \begin{bmatrix} p^x_k \; v^x_k \; p^y_k \; v^y_k \; \omega_k \end{bmatrix}^T, \quad w_k \sim \mathcal{N}[0, \; Q_2],
\]

\[
F_k = \begin{bmatrix} 1 & \sin \omega_{k-1} T & 0 & -\cos \omega_{k-1} T & 0 \\ 0 & \cos \omega_{k-1} T & 0 & -\sin \omega_{k-1} T & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
H_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad v_k \sim \mathcal{N}[0, \; R]
\]

where \( T \) is the sampling period and \( Q_2 \) is [19]

\[
Q_2 = T \cdot \begin{bmatrix} 0_{4 \times 4} & 0_{4 \times 1} \\ 0_{1 \times 4} & 0 \end{bmatrix}
\]

This formulation was chosen because it is simple and it has been tested that the result is rather insensitive to this simplification [19]. In the simulation, the following parameters were adopted: \( R = 20^2 \) m\(^2\), \( Q_2 = 10^{-12} \text{diag}([0, 0, 0, 0, 1]) \) (rad/s)\(^2\) (\( Q_2 = 0 \) for the true trajectories), \( R_L = 300 \) m, \( R_U = 305 \) m, \( [p^x_c, p^y_c] = [0, 302.5] \) m. Three filters based on this nonlinear model were compared:

1) UF without considering the constraints.
2) UF with the unconstrained estimated mean projected (UF\_Proj), in which \( W_k = P_{k|k}^{-1} \).
3) UF with our approach (with modified covariance) for projection (UF\_Proj\_MD), in which \( W_k = P_{k|k}^{-1} \), \( \lambda = 0, 0.5, 1 \) were selected for further discussions. The filter with \( \lambda = 1 \) was found to diverge in this scenario.

The above filters were initialized optimally based on the first two measurements using a predicted turn rate (e.g., \( \omega_0 = 0 \) for simplicity). Since the constraints are one-dimensional, the closed form (16) was adopted for projection in KF\_Proj, KF\_Proj\_MD, UF\_Proj and UF\_Proj\_MD.

The comparison results are the RMSE (root mean square errors) over 300 Monte Carlo runs for the two scenarios, shown in the following figures.

![Figure 3. RMS errors of position (m) in S1](image_url)

There is little difference for KF\_Proj\_MD with different \( \lambda \) values.
Generally, the simulation results demonstrate that the proposed projection approach based on UT is effective, and that utilizing the constraint information based on the projection approaches is effective to improve the estimation performance. In S1, the $\text{KF}_{\text{Proj}}$ filters ($\lambda = 0, 0.5, 1$) outperform the other filters. The performance of $\text{KF}_{\text{Proj}}$ is insensitive to different $\lambda$ values (with linear inequality constraints). In S2, however, the performance of $\text{UF}_{\text{Proj}}$ is rather sensitive to $\lambda$. The projection approach based on UT without covariance modification ($\lambda = 1$) was found to diverge in S2. The corresponding results are not shown. Simulation results also show that $\text{UF}_{\text{Proj}}$ with $\lambda = 0.5$ and $\lambda = 0$ has the best and worst performance respectively (discussions on $\lambda$ are given in Remark 2). The comparison results illustrate that modifying the constrained covariance by selecting $\lambda$ is important for UT-based estimation with nonlinear inequality constraints as in S2, but not for linear inequality constraints as in S1. Also, the proposed UT-based projection approach with covariance modification appears effective in incorporating the constraint information for state estimation. Further research will be focused on the determination of $\lambda$ for typical practical applications.

IV. CONCLUSIONS

A projection method for nonlinear inequality constraints has been formulated as a special nonlinear function. Based on this formulation, an approach has been proposed to relieve the computational burden, and an analytic solution for one-dimensional constraints has been derived. The computational complexity is largely reduced by this approach. To solve the problem for which the covariance matrix may be ill-conditioned while using UT for projection with hard constraints, a UT-based projection approach with a modified covariance matrix has been proposed, along with discussions on the connections between this approach and some other approaches. Two scenarios for ground moving target tracking with linear and nonlinear inequality constraints have been simulated to evaluate the proposed approaches. Simulation results demonstrate that the proposed approach appears effective.

REFERENCES


