A Technique for Deriving Multitarget Intensity Filters Using Ordinary Derivatives

ROY STREIT

In multitarget tracking problems based on finite point process models of targets and measurements, it is known that the distribution of the Bayes posterior point process is a ratio of functional derivatives of a joint probability generating functional. It is shown here that these functional derivatives can be found by evaluating ordinary derivatives. The method is exact, not approximate. Several examples are presented, including multisensor target tracking and extended-target tracking. The method is well suited to the needs of particle filter implementations.

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Author's addresses: Metron, Inc., 1818 Library St., Reston, VA 20109.

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1. INTRODUCTION

This paper shows that functional derivatives of the probability generating functional (PGFL) of a finite point process can be calculated using ordinary derivatives. The result is new, and it is potentially useful to the class of Bayesian multitarget tracking problems that is based on finite point process models for targets and measurements. In this class, the distribution of the Bayes posterior multitarget process is a ratio of functional derivatives of the joint measurement-target PGFL. In some problems evaluating the functional derivatives is only a tedious task, but in other problems the number of terms in the derivatives is prohibitively large and limits practical applications of the method.

The proof is straightforward—we reduce the PGFL to an ordinary function that is conceptually straightforward to differentiate. This function is called a secular function to emphasize that it is an "ordinary" function and not a functional. Existing symbolic software packages can be used to differentiate the secular function, a fact that is potentially of practical importance since software for functional differentiation of the PGFL does not seem to be available. The methods of this paper use the established theory of PGFLs and their functional derivatives.

The proposed methods are compatible with particle, or sequential Monte Carlo (SMC), filter implementations. The basic strategy is to embed symbolic differentiation software in the production code and evaluate the symbolic derivatives of the secular functions at the points of the particle filter. One of the purposes of this paper is to show that this is a theoretically feasible strategy. Its practical utility is outside the scope of the paper.

Two tracking applications where functional differentiation causes serious difficulties are discussed. One is multisensor target tracking [9]. The other is extended-target tracking problems in which targets can produce more than one measurement [8, 11]. The secular functions for both problems are derived.

Functional differentiation of the PGFL is the result of a double limit. A theoretical question naturally arises, "Can these limits be interchanged?" The answer is, "Yes, for the problems of interest here." This result seems to be new. It gives a better understanding of PGFLs and their relationship to classical probability generating functions (PGFs).

Section II speaks of the PGFL as an encoding of the multitarget tracking problem and functional differentiation of the PGFL as the decoding algorithm. Section III gives a simple example of the method we use to reduce PGFLs to secular functions. Section IV proves that for the class of PGFLs of interest in this paper, ordinary derivatives of secular functions are *identical* to functional derivatives of PGFLs. Section V gives several examples of secular functions, including those for multisensor and extended-target tracking problems. Section VI discusses finite differences and series expansion methods for approximating secular functions. Section VII gives conclusions.

2. FINITE POINT PROCESSES AND PGFLS

The grand canonical ensemble is defined to be the set of all finite lists of points in a given space. A random variable whose outcomes are in this ensemble is called a finite point process. When the space is continuous (i.e., it has no discrete elements) and the probability distribution is orderly (i.e., the underlying Borel measure has no atoms), realizations of the finite point process cannot have repeated elements, that is, the outcomes of the finite point process are sets with probability one.

Some familiarity is assumed with the definitions and properties of finite point processes such as given in [1, 3, 10, 12]. Such familiarity should include PGFLs, which are defined in [10] as an expectation of a random product with respect to a probability distribution over the grand canonical ensemble. Finally, readers are assumed to be familiar with the application of PGFLs to multitarget tracking, for which see [7] or the comparison paper [13].

In applications of finite point processes to Bayesian multitarget tracking, the joint measurement-target PGFL "encodes" the over-all probability structure of the problem. The measurement-to-target/clutter assignments are assumed unknown in this paper, so the probabilistic structure is inherently combinatorial. The PGFL of the Bayes posterior process is derived from the joint PGFL by conditioning on the measurement set.

To find the Bayes posterior probability of a given event, it is necessary to "decode" its PGFL. Decoding is equivalent to functional differentiation of the PGFL. Differentiation is straightforward in principle, but it often has high computational complexity because large numbers of different kinds of terms can appear in the functional derivatives. Decoding the PGFL of the Bayes posterior point process is the problem of interest in this paper.

The primary purpose of the paper is to show that functional derivatives of the PGFL are equivalent to ordinary derivatives of functions that are easily derived from the PGFL. These functions are of independent interest, so we refer to them as secular functions. Several examples are given in Section V. The number of terms in their derivatives is prohibitively large, so the symbolic derivatives are not shown; however, the derivatives can be found using widely available software. Many software packages can be configured to evaluate the symbolic derivative numerically at specified points—thus avoiding the need for manually recoding. The secular method is exact, not approximate.

For particle tracking filter implementations, we need to evaluate the functional derivative of the PGFL for every particle (point) in the current particle set. These numerical values are used to update the particle weights, and they are subsequently used to resample the particles in the SMC update step. The same particle weights can also be computed by evaluating the symbolic derivatives of secular functions. This procedure is potentially important for applications in which derivatives are too difficult to find by hand; however, further discussion is outside the scope of the paper.

3. ILLUSTRATIVE EXAMPLE

Secular functions are obtained from PGFLs by a straightforward procedure. In this section we illustrate the technique with an example not unlike what is encountered in tracking applications.

The PGF of a Poisson distributed random number $N \geq 0$ with mean $\lambda \geq 0$ is given by $F(s) = \exp(-\lambda + \lambda s)$, where s is a complex-valued variable. Define the functional $I[g] = \int_Y g(y)q(y)dy$, where Y is a closed and bounded subset of $\mathbb{R}^2$ and $q(\cdot)$ is a continuously differentiable probability density function (PDF) in the interior of Y. The function $g(\cdot)$ is assumed to be such that $|g(y)| \leq 1$ for all $y \in Y$ and infinitely differentiable in the interior of Y. Define the functional

$$\Psi[g] = F(I[g]) = \exp\left(-\lambda + \lambda \int_Y g(y)q(y)dy\right). \quad (1)$$

Note that $\Psi[g]$ is the PGFL of the nonhomogeneous Poisson point process on Y whose intensity function is $\lambda q(\cdot)$. Using the Calculus of Variations, let the variation $\gamma(\cdot)$ be a bounded function on Y and infinitely differentiable in the interior of Y. The functional derivative of $\Psi[g]$ with respect to the variation $\gamma(\cdot)$ is defined by

$$\frac{\partial \Psi}{\partial \gamma}[g] \equiv \left.\frac{d\Psi[g+\alpha\gamma]}{d\alpha}\right|_{\alpha=0} = \lambda\left(\int_Y \gamma(y)q(y)dy\right)\exp\left(-\lambda + \lambda\int_Y g(y)q(y)dy\right). \quad (2)$$

Ordinary derivatives are denoted by "d" to distinguish them from functional derivatives which are denoted by "∂." Let c be an interior point of Y. Denote the Dirac delta function at c by $\delta_c(y)$. We will often refer to $\delta_c(y)$ as an "impulse" at the point c. Let $\gamma_n^c(y)$, $n = 1,2,\ldots$, denote a sequence of test functions for $\delta_c(y)$. (General discussions of test functions are widely available; see the classic text [6].) There are many possible choices for $\gamma_n^c(y)$. To be specific, we take $\gamma_n^c(y)$ to be the PDF of a bivariate Gaussian random variable with mean c and covariance matrix equal to the identity matrix scaled by n^{-2} that is truncated and normalized to integrate to one on Y. Thus, $\gamma_n^c(y)$ is non-negative, infinitely differentiable interior to Y, and is unimodal with a maximum value occurring at the point c. The sequence itself is not bounded. For such test sequences it is easy to prove that

$$\lim_{n\to\infty}\int_Y q(y)\gamma_n^c(y)dy = q(c). \quad (3)$$

Note that the integral in (3) is evaluated before taking the limit. For each n, it follows from (2) that

$$\frac{\partial \Psi}{\partial \gamma_n^c}[g] = \lambda \left(\int_Y \gamma_n^c(y) q(y) dy \right) \exp\left(-\lambda + \lambda \int_Y g(y) q(y) dy \right). \tag{4}$$

The functional derivative of the PGFL $\Psi[g]$ with respect to an impulse at $c \in Y$ is the limit of (4) as $n \to \infty$. Thus, using (3),

$$\frac{\partial \Psi}{\partial c}[g] \equiv \lim_{n\to\infty} \frac{\partial \Psi}{\partial \gamma_n^c}[g] = \lambda q(c) \exp\left(-\lambda + \lambda \int_Y g(y) q(y) dy \right). \tag{5}$$

Using (1), we define

$$\begin{aligned} J(\alpha) &= \lim_{n\to\infty} \Psi[g + \alpha \gamma_n^c] \\ &= \exp\left(-\lambda + \lambda \int_Y g(y) q(y) dy + \lambda \alpha q(c) \right). \end{aligned} \tag{6}$$

The ordinary derivative $J'(0)$ is identical to (5). In this paper we call $J(\cdot)$ a secular function.

The example shows that the functional derivative of the PGFL (1) at c is identical to the derivative of its secular function (6) at zero. The next section shows that the technique extends to more general problems.

4. SECULAR FUNCTIONS

Our goal is to present results of the kind needed for tracking applications, not to give a general mathematical treatment. We assume that $F(s) = \sum_{n=0}^{\infty} \Pr\{n\} s^n$ is the PGF of a discrete random variable with outcomes in the non-negative integers, $\mathbb{N}$. Thus, $F(s)$ is analytic at the origin in the complex s-plane, $\mathbb{C}$. Because $F(1) = 1$, it is analytic in a region that includes the closed unit disc. Let Y be a closed and bounded subset of the Euclidean space $\mathbb{R}^d$, $d \geq 1$. The function $q : Y \to \mathbb{R}$ is assumed to be a PDF on Y and continuously differentiable at interior points of Y except possibly for jump discontinuities of the kind that occur, e.g., by truncating a Gaussian distribution. The function $g : Y \to \mathbb{R}$ is assumed to be such that $|g(y)| \leq 1$ for all $y \in Y$. The functional $\Psi[g]$ is defined as in the example, that is,

$$\Psi[g] = F\left(\int_Y g(y) q(y) dy \right). \tag{7}$$

Let the variation $\gamma : Y \to \mathbb{R}$ be bounded, i.e., for some number B, $|\gamma(y)| \leq B < \infty$ for all $y \in Y$. Then $\Psi[g + \alpha\gamma]$, considered as a function of the complex variable α, is analytic in an open neighborhood of the origin, i.e., in the open disc $|\alpha| < r$ where r is sufficiently small. The functional derivative of $\Psi[g]$ with respect to γ is defined by

$$\frac{\partial \Psi}{\partial \gamma}[g] = \left. \frac{d\Psi[g + \alpha\gamma]}{d\alpha} \right|_{\alpha=0}. \tag{8}$$

Substituting (7) into (8) gives, by direct calculation,

$$\frac{\partial \Psi}{\partial \gamma}[g] = F^{(1)}\left(\int_Y g(y) q(y) dy \right) \int_Y q(y) \gamma(y) dy, \tag{9}$$

where $F^{(1)}(\cdot)$ denotes the ordinary first derivative of $F(\cdot)$.

We seek the functional derivative of $\Psi[g]$ with respect to an impulse at the point $c \in Y$. As in the example, let $\gamma_n^c(y)$, $n = 1, 2, \ldots$, denote a sequence of test functions for the Dirac delta function $\delta_c(y)$ at $c \in Y$. Test functions are bounded, so the functional derivative of $\Psi[g]$ with respect to a given test function is well-defined. The functional derivative of $\Psi[g]$ with respect to an impulse at c is defined by the limit

$$\begin{aligned} \frac{\partial \Psi}{\partial c}[g] &\equiv \lim_{n\to\infty} \frac{\partial \Psi}{\partial \gamma_n^c}[g] \\ &= F^{(1)}\left(\int_Y g(y) q(y) dy \right) \lim_{n\to\infty} \int_Y q(y) \gamma_n^c(y) dy \\ &= F^{(1)}\left(\int_Y h(y) q(y) dy \right) q(c). \end{aligned} \tag{10}$$

Using the same test sequence, the secular function corresponding to $\Psi[g]$ is defined by

$$\begin{aligned} J(\alpha; c) &= \lim_{n\to\infty} \Psi[g + \alpha \gamma_n^c] \\ &= \lim_{n\to\infty} F\left(\int_Y g(y) q(y) dy + \alpha \int_Y \gamma_n^c(y) q(y) dy \right) \\ &= F\left(\int_Y g(y) q(y) dy + \alpha q(c) \right). \end{aligned} \tag{11}$$

Taking the limit inside the argument of $F(\cdot)$ is justified by analyticity. Note that J depends implicitly on g. The derivative of J with respect to α evaluated at zero is

$$\frac{dJ}{d\alpha}(0; c) = \left. \frac{d}{d\alpha} J(\alpha; c) \right|_{\alpha=0} = F^{(1)}\left(\int_Y g(y) q(y) dy \right) q(c). \tag{12}$$

Comparing (12) and (10) shows that

$$\frac{\partial \Psi}{\partial c}[g] = \frac{dJ}{d\alpha}(0; c). \tag{13}$$

Thus, the functional derivative of Ψ is identical to the ordinary derivative of its secular function J.

The result (13) does not alter the theory of PGFLs and their functional derivatives. It does, however, show that functional derivatives of the PGFL can be replaced by ordinary derivatives of the secular function. This means that many functions of interest in tracking applications (e.g., intensity and pair-correlation) can be found by differentiating the secular function of the PGFL.

The functional derivative of the PGFL is a double limit. The above results show that the order in which these limits are taken can be interchanged. Informally,

$$\begin{aligned}\lim_{n\to\infty}\lim_{\alpha\to 0}&\frac{d\Psi[g+\alpha\gamma_n^c]}{d\alpha}\\&=\frac{\partial\Psi}{\partial c}[g]\\&=\frac{dJ}{d\alpha}(0;c)=\lim_{\alpha\to 0}\lim_{n\to\infty}\frac{d\Psi[g+\alpha\gamma_n^c]}{d\alpha}.\end{aligned}\quad(14)$$

4.1. Extensions for Multivariate PGFs and Cross-Derivatives

The basic result (13) extends to more general functions and functionals. Suppose that

$$\Psi[g]=F\left(\int_Y g(y)q_1(y)dy,\ldots,\int_Y g(y)q_k(y)dy\right),\quad(15)$$

where $F(s_1,\ldots,s_k)$, $k\geq 1$, is the multivariate PGF of a random vector of integers with outcomes in $\mathbb{N}^k$. Hence, $F(\cdot)$ is analytic at the origin $\mathbf{0}=(0,\ldots,0)\in\mathbb{C}^k$. The functions $q_i: Y\to\mathbb{R}$, $i=1,\ldots,k$, are assumed to be continuously differentiable PDFs on Y except possibly for jump discontinuities. As before, let c be interior to Y, and let $\gamma_n^c(y)$, $n=1,2,\ldots$, be a test sequence for $\delta_c(y)$. The secular function of $\Psi[g]$ is defined to be

$$\begin{aligned}J(\alpha;c)&=\lim_{n\to\infty}\Psi[g+\alpha\gamma_n^c]\\&=\lim_{n\to\infty}F\left(\int_Y(g(y)+\alpha\gamma_n^c(y))q_1(y)dy,\ldots,\right.\\&\qquad\left.\int_Y(g(y)+\alpha\gamma_n^c(y))q_k(y)dy\right)\\&=F\left(\int_Y g(y)q_1(y)dy+\alpha q_1(c),\ldots,\right.\\&\qquad\left.\int_Y g(y)q_k(y)dy+\alpha q_k(c)\right).\end{aligned}$$

The ordinary derivative is

$$\frac{dJ}{d\alpha}(0;c)=\left.\frac{dJ}{d\alpha}\right|_{\alpha=0}=\sum_{\ell=1}^k F_\ell^{(1)}\left(\int_Y g(y)q_1(y)dy,\ldots,\int_Y g(y)q_k(y)dy\right)q_\ell(c),\quad(16)$$

where $F_\ell^{(1)}(\cdot)$ denotes the (ordinary) first derivative of $F(\cdot)$ with respect to argument ℓ. Direct calculation of the functional derivative of $\Psi[g]$ with respect to an impulse at c shows that it is identical to the right hand side of (16). Thus, (14) holds for PGFLs of the form (15).

The functional derivative with respect to impulses at points $\mathbf{y}=\{y_1,\ldots,y_m\}$, $m\geq 0$, is defined by

$$\Psi_{\mathbf{y}}[g]\equiv\frac{\partial^m}{\partial y_1\cdots\partial y_m}\Psi[g].\quad(17)$$

The points $y_i\in Y$ are assumed distinct. Let $\gamma_n^{y_i}(y)$, $n=1,2,\ldots$, denote a test function sequence for the Dirac delta function at y_i. The secular function of (15) is defined (cf. [10, Eqn. (4.11)] and (26)–(27) below) by substituting a test sequence for a weighted train of Dirac delta functions:

$$\begin{aligned}J(\boldsymbol{\alpha};\mathbf{y})&=\lim_{n\to\infty}\Psi\left[g(y)+\sum_{i=1}^m\alpha_i\gamma_n^{y_i}(y)\right]\\&=F\left(\int_Y g(y)q_1(y)dy+\sum_{i=1}^m\alpha_i q_1(y_i),\ldots,\right.\\&\qquad\left.\int_Y g(y)q_k(y)dy+\sum_{i=1}^m\alpha_i q_k(y_i)\right),\end{aligned}\quad(18)$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)^T\in\mathbb{R}^m$. The secular function is thus a function of the coefficient vector $\boldsymbol{\alpha}$ and depends implicitly on the function g. It is easily seen that

$$\Psi_{\mathbf{y}}[g]=J_{\boldsymbol{\alpha}}(\mathbf{0};\mathbf{y})\equiv\left.\frac{d^m}{d\alpha_1\cdots d\alpha_m}J(\boldsymbol{\alpha};\mathbf{y})\right|_{\alpha_1=\cdots=\alpha_m=0},\quad(19)$$

where $\mathbf{0}$ denotes the zero vector. Thus, the functional derivative of Ψ with respect to impulses at the points $\mathbf{y}\equiv\{y_1,\ldots,y_m\}$ is identical to the first order mixed derivative of the secular function. Such derivatives are called cross-derivatives in the automatic differentiation literature [5]. See Section VI for further comment on this topic.

4.2. Secular Functions for Multivariate PGFLs

Joint PGFLs correspond to two or more finite point processes defined on possibly different spaces. In tracking applications, for example, the joint PGFL can be the joint measurement-target process on the measurement space Y and the target space S. The discussion here is limited to these processes. The PGFL $\Psi[g,h]$ is assumed known. The extension to more than two processes is straightforward.

Conditioned on the (distinct) measurements $\mathbf{y}=\{y_1,\ldots,y_m\}$, $m\geq 0$, the PGFL of the Bayes posterior point process is the normalized functional derivative:

$$\Psi[h\mid\mathbf{y}]\equiv\frac{\Psi_{\mathbf{y}}[g,h]|_{g(\cdot)=0}}{\Psi_{\mathbf{y}}[g,1]|_{g(\cdot)=0}}=\frac{\Psi_{\mathbf{y}}[0,h]}{\Psi_{\mathbf{y}}[0,1]},\quad(20)$$

where the functional derivative with respect to impulses at the points of $\mathbf{y}$ is

$$\Psi_{\mathbf{y}}[g,h]\equiv\frac{\partial^m}{\partial y_1\cdots\partial y_m}\Psi[g,h].\quad(21)$$

As a check, note that $\Psi[1\mid\mathbf{y}]=1$ for all $\mathbf{y}$. The nth factorial moment of (20) is defined to be the functional derivative with respect to impulses at the (distinct) points $\mathbf{x}=\{x_1,\ldots,x_n\}$, $n\geq 0$; explicitly,

$$\begin{aligned}m_{[n]}(x_1,\ldots,x_n)&=\Psi_{\mathbf{x}}[h\mid\mathbf{y}]|_{h(\cdot)=1}\equiv\frac{\partial^n}{\partial x_1\cdots\partial x_n}\Psi[1\mid\mathbf{y}]\\&=\frac{\Psi_{\mathbf{yx}}[0,1]}{\Psi_{\mathbf{y}}[0,1]},\end{aligned}\quad(22)$$

where

$$\Psi_{\mathbf{yx}}[g,h] \equiv \frac{\partial^m}{\partial y_1 \cdots \partial y_m} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \Psi[g,h]. \tag{23}$$

Note that the nth factorial moment is a function of the point $\mathbf{x}$. For a careful definition of factorial moments, see [10], the definitive text [3], or the recent paper [13].

The first factorial moment is commonly known as the intensity function of the point process. In tracking applications it is sometimes called the probability hypothesis density (PHD). For $n = 1$, (22) can be written in an interesting logarithmic form as

$$m_{[1]}(x_1) = \frac{\partial}{\partial x_1} \log \Psi[h \mid \mathbf{y}] \Big|_{h(\cdot)=1}. \tag{24}$$

Intuitively, the second factorial moment is a "two point" intensity function. The pair correlation function for $x_1 \neq x_2$ is defined as the ratio

$$\rho(x_1,x_2) = \frac{m_{[2]}(x_1,x_2)}{m_{[1]}(x_1) m_{[1]}(x_2)}. \tag{25}$$

From the independent sampling property of PPPs it can be shown that $m_{[2]}(x_1,x_2) = m_{[1]}(x_1)m_{[1]}(x_2)$, so that $\rho(x_1,x_2) = 1$. A point process is said to be attractive if $\rho(x_1,x_2) > 1$ for all distinct points, and repulsive if $\rho(x_1,x_2) < 1$.

The derivative of the PGFL is evaluated for the constant functions $g(\cdot) = 0$ and $h(\cdot) = 1$ to find, respectively, the event probabilities and factorial moments. For this reason we employ the simultaneous perturbations (see [10, Eqn. (4.11)])

$$g(y) = \sum_{i=1}^{m} \alpha_i \delta_{y_i}(y), \quad y \in Y, \tag{26}$$

$$h(s) = 1 + \sum_{j=1}^{n} \beta_j \delta_{x_j}(s), \quad s \in S, \tag{27}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$. The sums are defined to be zero for $m = 0$ and $n = 0$. Therefore, the secular function corresponding to the joint PGFL is

$$J(\boldsymbol{\alpha},\boldsymbol{\beta};\mathbf{y},\mathbf{x}) = \Psi\left[\sum_{i=1}^{m} \alpha_i \delta_{y_i}(y), 1 + \sum_{j=1}^{n} \beta_j \delta_{x_j}(s)\right]. \tag{28}$$

The test function sequence forms of (26)–(28) are somewhat tedious, so we do not use them. (The result is unchanged by using test sequence versions of these expressions.) The methods of the previous subsection show that the ordinary derivatives of J are identical to the functional derivatives of Ψ; explicitly,

$$J_{\boldsymbol{\alpha\beta}}(\boldsymbol{\alpha},\boldsymbol{\beta};\mathbf{y},\mathbf{x})|_{\boldsymbol{\alpha}=\mathbf{0},\boldsymbol{\beta}=\mathbf{0}} \equiv \Psi_{\mathbf{yx}}[g,h]|_{g(\cdot)=0,h(\cdot)=1}. \tag{29}$$

Therefore, functional derivatives of the joint PGFL can be replaced wherever they occur by ordinary derivatives of its secular function.

Of particular interest is the first factorial moment, or intensity, of the Bayes posterior point process. This is the special case $n = 1$ of (24). Written in terms of the secular function, with $\boldsymbol{\beta} = \beta_1$ and $\mathbf{x} = x_1$,

$$m_{[1]}(x_1) = \frac{d}{d\beta_1} \log J_{\boldsymbol{\alpha}}(\mathbf{0},\beta_1;\mathbf{y},x_1)\Big|_{\beta_1=0}. \tag{30}$$

Recall that we evaluate the derivative at $\beta_1 = 0$ because (27) is a perturbation about $h(s) = 1$. The second factorial moment of the Bayes posterior process is, from (22) with $n = 2$,

$$m_{[2]}(x_1,x_2) \equiv \frac{1}{J_{\boldsymbol{\alpha}}(\mathbf{0},\mathbf{0};\mathbf{y},\mathbf{x})} \frac{d^2}{d\beta_1 d\beta_2} J_{\boldsymbol{\alpha}}(\mathbf{0},\boldsymbol{\beta};\mathbf{y},\mathbf{x})\Big|_{\beta_1=\beta_2=0}, \tag{31}$$

where $\boldsymbol{\beta} = (\beta_1,\beta_2)$ and $\mathbf{x} = (x_1,x_2)$.

5. EXAMPLES OF SECULAR FUNCTIONS

Example 1 starts with the joint PGFL of the single sensor point target problem. The PHD intensity filter is derived from the secular function of the PGFL of the Bayes posterior point process. Time indexing in this and the other examples is suppressed to simplify notation. The pair-correlation function is derived in Example 2. Secular functions have little to offer in these examples because functional derivatives can be evaluated by hand. The joint PGFLs for multisensor target tracking and extended-target tracking are given in Examples 3 and 4, respectively. The corresponding secular functions are then derived. The functional derivatives cannot be found by hand for these examples but the ordinary derivatives of the secular functions can be found using reliable and efficient software. These expressions are identical to the functional derivatives of the PGFL.

5.1. Example 1: Secular Functions for the PHD/Intensity Filter

The Bayes posterior point process is not a Poisson point process (PPP). To close the Bayesian recursion, the posterior process is approximated [7] by a PPP whose intensity function is matched to the intensity, or first moment, of the Bayes posterior process. Predicting the intensity of this PPP approximation forward to the current time gives a PPP with intensity $f(s)$. After the prediction step, the joint PGFL of the measurement-target process is

$$\begin{aligned}\Psi[g,h] = \exp\Big[&-\int_Y \lambda(y)dy + \int_Y g(y)\lambda(y)dy - \int_S f(s)ds \\ &+ \int_S h(s)f(s)ds - \int_S h(s)P^D(s)f(s)ds \\ &+ \int_S \int_Y g(y)h(s)p(y \mid s)P^D(s)f(s)dy\,ds\Big],\end{aligned} \tag{32}$$

where Y and S are the sensor measurement and target state spaces, respectively, $\lambda(y)$ is the intensity function of a PPP model for sensor clutter measurements, $P^D(s)$ is the probability of detecting a target in state s, and $p(y \mid s)$ is the sensor measurement likelihood function. The PGFL is defined for bounded functions $g: Y \to \mathbb{R}$ and $h: S \to \mathbb{R}$. The PGFL (32) depends on many assumptions about target motion, target measurement, the clutter process, and the measurement-to-target assignments, to name only a few. These assumptions and the derivation of the PGFL are not given here because we take the joint PGFL as our starting point. Further details can be found in [7] and also in [13].

The secular function is defined by (28). In the case of the PGFL (32), this gives

$$\begin{aligned} J(\boldsymbol{\alpha},\boldsymbol{\beta};\mathbf{y},\mathbf{x}) &= G_0 \exp\left[\sum_{i=1}^{m} \alpha_i \left(\lambda(y_i) + \int_S p(y_i \mid s) P^D(s) f(s) ds\right)\right. \\ &\quad + \sum_{j=1}^{n} \beta_j (1 - P^D(x_j)) f(x_j) \\ &\quad \left. + \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i \beta_j p(y_i \mid x_j) P^D(x_j) f(x_j)\right], \end{aligned} \tag{33}$$

where

$$G_0 = \exp\left[-\int_Y \lambda(y) dy - \int_S f(s) P^D(s) ds\right]. \tag{34}$$

Note that $J(\cdot)$ is the exponential of a quadratic polynomial in the components of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The derivatives of $J(\cdot)$ are straightforward to compute by hand when $\min\{m,n\}$ is small.

The PGFL of the Bayes posterior point process is, using (20),

$$\Psi[h] = \frac{\Psi_{\mathbf{y}}[0,h]}{\Psi_{\mathbf{y}}[0,1]}. \tag{35}$$

The intensity function of this process is $m_{[1]}(x_1)$. It is given in terms of the secular function by (30), where the derivative $J_\alpha(\cdot)$ is, from (33) with $n = 1$,

$$\begin{aligned} J_\alpha(\mathbf{0},\beta_1;\mathbf{y},x_1) &= J(\mathbf{0},\beta_1;\mathbf{y},x_1) \\ &\quad \times \prod_{i=1}^{m} \left(\lambda(y_i) + \int_S h(s) p(y_i \mid s) P^D(s) f(s) ds \right. \\ &\quad \left. + \beta_1 p(y_i \mid x_1) P^D(x_1) f(x_1)\right). \end{aligned} \tag{36}$$

The derivative of the logarithm of (36) with respect to β_1 evaluated at $\beta_1 = 0$ is

$$\begin{aligned} m_{[1]}(x_1) &= \frac{1}{J_\alpha(\mathbf{0},0;\mathbf{y},x_1)} \frac{d}{d\beta_1} J_\alpha(\mathbf{0},0;\mathbf{y},x_1) \\ &= (1 - P^D(x_1)) f(x_1) + \sum_{i=1}^{m} \frac{p(y_i \mid x_1) P^D(x_1) f(x_1)}{\lambda(y_i) + \int_S p(y_i \mid s) P^D(s) f(s) ds}. \end{aligned} \tag{37}$$

The expression (37) is the PHD filter information update.

5.2. Example 2. Pair-Correlation Function of the Bayes Posterior Target Process

"Spooky action at a distance" [4] is a source of concern using point process models for tracking independent targets. One cause is nontrivial pair-correlation [1] in the Bayes posterior process. It is shown in this example that the Bayes posterior target process is repulsive for all x_1 and x_2. This result was first derived in [2].

For Example 1 the second factorial moment is given by the normalized second derivative (31) of the secular function. The derivative $J_\alpha(\cdot)$ is, using (33) with $n = 2$,

$$\begin{aligned} J_\alpha(\mathbf{0},\boldsymbol{\beta};\mathbf{y},\mathbf{x}) &= J(\mathbf{0},\boldsymbol{\beta};\mathbf{y},\mathbf{x}) \prod_{i=1}^{m} \left(\lambda(y_i) + \int_S h(s) p(y_i \mid s) P^D(s) f(s) ds \right. \\ &\quad \left. + \sum_{j=1}^{2} \beta_j p(y_i \mid x_j) P^D(x_j) f(x_j)\right). \end{aligned} \tag{38}$$

The first derivative of the logarithm of (38) with respect to β_ℓ, $\ell = 1,2$, is

$$\begin{aligned} &\frac{1}{J_\alpha(\mathbf{0},\boldsymbol{\beta};\mathbf{y},\mathbf{x})} \frac{d}{d\beta_\ell} J_\alpha(\mathbf{0},\boldsymbol{\beta};\mathbf{y},\mathbf{x}) \\ &\quad = (1 - P^D(x_\ell)) f(x_\ell) + \sum_{i=1}^{m} \frac{p(y_i \mid x_\ell) P^D(x_\ell) f(x_\ell)}{\lambda(y_i) + \int_S p(y_i \mid s) P^D(s) f(s) ds + \sum_{j=1}^{2} \beta_j p(y_i \mid x_j) P^D(x_j) f(x_j)}. \end{aligned} \tag{39}$$

Now let $\ell = 1$ in (39) and take the first derivative with respect to β_2. Setting $\beta_1 = \beta_2 = 0$, rearranging terms, and substituting the first moment (37) evaluated at the points x_1 and x_2 of $\mathbf{x}$ gives

$$\frac{1}{J_\alpha(\mathbf{0},\mathbf{0};\mathbf{y},\mathbf{x})}\frac{d^2}{d\beta_1 d\beta_2}J_\alpha(\mathbf{0},\mathbf{0};\mathbf{y},\mathbf{x})$$
$$= m_{[1]}(x_1)m_{[1]}(x_2) - \sum_{i=1}^{m}\frac{p(y_i \mid x_1)P^D(x_1)f(x_1)p(y_i \mid x_2)P^D(x_2)f(x_2)}{\left(\lambda(y_i) + \int_S p(y_i \mid s)P^D(s)f(s)ds\right)^2}. \tag{40}$$

From (31) the left hand side of (40) is seen to be the second factorial moment of the PGFL (35) of the Bayes posterior point process. Dividing by the product of first factorial moments gives

$$\rho(x_1,x_2) = 1 - \frac{1}{m_{[1]}(x_1)m_{[1]}(x_2)} \times \sum_{i=1}^{m}\frac{p(y_i \mid s_1)P^D(s_1)f(s_1)p(y_i \mid s_2)P^D(s_2)f(s_2)}{\left(\lambda(y_i) + \int_S p(y_i \mid s)P^D(s)f(s)ds\right)^2}. \tag{41}$$

It is evident that $\rho(x_1,x_2) < 1$; therefore, the Bayes posterior process is repulsive.

5.3. Example 3. Multisensor Multitarget Tracking Filters

Let $L \geq 1$ sensors produce conditionally independent measurements in the spaces Y^ℓ, $\ell = 1,\ldots,L$. Example 1 is the special case $L = 1$. The joint PGFL [9] can be written explicitly as

$$\Psi[g^1,\ldots,g^L,h] = \exp\left(-\sum_{\ell=1}^{L}\int_{Y^\ell}\lambda^\ell(y)dy + \sum_{\ell=1}^{L}\int_{Y^\ell}g^\ell(y)\lambda^\ell(y)dy - \int_S f(s)ds + \int_S h(s)f(s)\prod_{\ell=1}^{L}\left(1 - P^{D^\ell}(s) + P^{D^\ell}(s)\int_{Y^\ell}g^\ell(y)p^\ell(y \mid s)dy\right)ds\right), \tag{42}$$

where the sensor likelihood functions and detection probabilities are given by $p^\ell(y \mid s)$ and $P^{D^\ell}(s)$, respectively.

Let $\boldsymbol{\alpha}^\ell = (\alpha_i^\ell : i = 1,\ldots,m^\ell)$, $\ell = 1,\ldots,L$. The secular function of (42) with respect to impulses at the sensor measurements $\mathbf{y}^\ell = \{y_i^\ell : i = 1,\ldots,m^\ell\}$, $y_i^\ell \in Y^\ell$, and at target state $\mathbf{x} = x_1$ is found by substituting

$$g^\ell(y) = \sum_{i=1}^{m^\ell}\alpha_i^\ell\delta_{y_i^\ell}(y), \quad y \in Y^\ell, \quad \ell = 1,\ldots,L,$$
$$h(s) = 1 + \beta_1\delta_{x_1}(s). \tag{43}$$

We do not discuss the pair-correlation function here, so the perturbation of h requires only one term. The logarithm of the secular function is

$$\log J(\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^L,\beta_1;\mathbf{y},x_1)$$
$$= -\sum_{\ell=1}^{L}\int_{Y^\ell}\lambda^\ell(y)dy + \sum_{\ell=1}^{L}\sum_{i=1}^{m^\ell}\alpha_i^\ell\lambda^\ell(y_i^\ell) - \int_S f(s)ds + \int_S f(s)\prod_{\ell=1}^{L}\left(1 - P^{D^\ell}(s) + P^{D^\ell}(s)\sum_{i=1}^{m^\ell}\alpha_i^\ell p^\ell(y_i^\ell \mid s)\right)ds + \beta_1 f(x_1)\prod_{\ell=1}^{L}\left(1 - P^{D^\ell}(x_1) + P^{D^\ell}(x_1)\sum_{i=1}^{m^\ell}\alpha_i^\ell p^\ell(y_i^\ell \mid x_1)\right), \tag{44}$$

where $\mathbf{y} = (\mathbf{y}^1,\ldots,\mathbf{y}^L)$. For $n = 1$ and $L = 1$, this expression is identical to (33).

Differentiating the secular function (44) with respect to $\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^L$ and evaluating it for $\alpha_i^\ell = 0$ gives the symbolic expression for the functional derivative at the point $x_1 \in S$. From (30), the symbolic derivative of the logarithm of this expression evaluated at $\beta_1 = 0$ gives the exact numerical value of the intensity function at the point x_1. Except for trivial cases, these derivatives are unsuited to manual differentiation because they have a prohibitively large number of terms. Symbolic differentiation packages can evaluate the required derivatives in principle, at least for sufficiently small problems. Nonetheless, regardless of the computer, these methods will struggle for larger problems. (The potential of automatic differentiation [5] to help in this problem remains to be studied.)

5.4. Example 4. Extended-Targets and Multiple Sensors

The joint PGFL for extended-targets and $L \geq 1$ sensors is

$$\Psi[g^1,\ldots,g^L,h] = \exp\left(-\sum_{\ell=1}^{L}\int_{Y^\ell}\lambda^\ell(y)dy + \sum_{\ell=1}^{L}\int_{Y^\ell}g^\ell(y)\lambda^\ell(y)dy - \int_S f(s)ds + \int_S h(s)f(s)\prod_{\ell=1}^{L}\Psi^{D\ell}[g^\ell \mid s]ds\right), \tag{45}$$

where $\Psi^{D\ell}[g^\ell \mid s]$ is the PGFL of the measurement point process for a target at $s \in S$. Let $G^\ell(z)$ denote the generating function of the number $\tau \geq 1$ of target measurements, conditioned on target detection. Thus $\Pr^\ell\{\tau = 0\} = 0$ and

$$G^\ell(z) = \sum_{\tau=1}^{\tau^\ell_{\max}} \Pr^\ell\{\tau\} z^\tau, \tag{46}$$

where z is the complex variable of the generating function and $\tau^\ell_{\max} \geq 1$ is the maximum number of measurements that a target at s can produce in sensor ℓ. Assuming that targets generate i.i.d. measurements in sensor ℓ with PDF $p^\ell(\cdot \mid s)$, the conditional measurement PGFL is

$$\Psi^{D\ell}[g^\ell \mid s] = 1 - P^{D\ell}(s) + P^{D\ell}(s) G^\ell\left(\int_{Y^\ell} g^\ell(y) p^\ell(y \mid s) dy\right). \tag{47}$$

For each $s \in S$, $\Psi^{D\ell}[1 \mid s] = 1$ since, by definition, $G^\ell(1) = 1$.

The special case of (45) for $L = 1$ and Poisson generating functions is discussed in [8] and the references therein. The PGFL for $L = 1$ using a more general target point process is given in [11].

Since targets can generate at most $\tau^\ell_{\max}$ measurements, $G^\ell(z)$ is a polynomial of degree $\tau^\ell_{\max}$. If $\tau^\ell_{\max} = 1$, targets generate at most one measurement with probability $P^{D^\ell}(s)$, so $G^\ell(z) \equiv z$ and (45) reduces to the PGFL (44) of Example 3. Another common model is that the number of measurements is Poisson distributed with mean μ_ℓ, conditioned on at least one measurement, so the generating function is $G^\ell(z) = (e^{\mu_\ell z} - 1)/(e^{\mu_\ell} - 1)$.

The joint PGFL for multiple sensors and multiple target measurements is determined by substituting (47) into (45). If only the intensity function is evaluated, the perturbation of h is limited to one term, and the secular function can be found using the same perturbations as (43). The secular function of $\Psi[g^1, \ldots, g^L, h]$ is then

$$\begin{aligned}\log J(\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^L, \beta_1; \mathbf{y}, x_1) = &-\sum_{\ell=1}^{L} \int_{Y^\ell} \lambda^\ell(y) dy + \sum_{\ell=1}^{L} \sum_{i=1}^{m^\ell} \alpha_i^\ell \lambda^\ell(y_i^\ell) - \int_S f(s) ds \\ &+ \int_S f(s) \prod_{\ell=1}^{L} \left(1 - P^{D\ell}(s) + P^{D\ell}(s) G^\ell\left(\sum_{i=1}^{m^\ell} \alpha_i^\ell p^\ell(y_i^\ell \mid s)\right)\right) ds \\ &+ \beta_1 f(x_1) \prod_{\ell=1}^{L} \left(1 - P^{D\ell}(x_1) + P^{D\ell}(x_1) G^\ell\left(\sum_{i=1}^{m^\ell} \alpha_i^\ell p^\ell(y_i^\ell \mid x_1)\right)\right).\end{aligned} \tag{48}$$

Derivatives of the secular function can be found by differentiating as needed under the integral sign (absolute convergence holds). As in Example 3, the numerical integrals can be calculated by summing the integrands over the current particle set.

6. DERIVATIVES OF SECULAR FUNCTIONS

The natural way to use secular functions in most applications is to find the exact symbolic derivatives using a software package for ordinary differentiation. Such software is often organized so that the symbolic derivative can be evaluated numerically at specified points, e.g., the particles in a particle filter, by exploiting the internal software representation of the derivative. This calculation bypasses the need to recode (or even to examine) the symbolic expressions.

Automatic differentiation (AD) methods are relatively new [5] techniques in which the numerical values of the symbolic derivative of a function are found without finding the symbolic derivative. These are exact methods (to machine precision), not approximations. Moreover, the additional computational effort is proportional to that of evaluating the function alone. AD methods are based on the chain rule. Their potential use for tracking applications is outside the scope of the present paper.

Alternatively, it may be worthwhile in some applications to consider classical numerical approximations of the symbolic derivatives. Two such methods, finite differences and Maclaurin series expansion, are briefly considered in this section for computing the intensity function.

6.1. Method 1: Classical Finite Differences

In the examples of Section V the derivatives of the secular function with respect to β_1 evaluated at $\beta_1 = 0$ can be evaluated easily for any $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^L) \in \mathbb{R}^M$, where $M = m^1 + \cdots + m^L$ is the total number of sensor measurements. Complexity grows with the number of derivatives with respect to $\boldsymbol{\alpha}$. The intensity function of the Bayes posterior point process is

$$m_{[1]}(x) = \frac{\left(\dfrac{d^M}{d\alpha_1 \cdots d\alpha_M} J'(\alpha, 0)\right)_{\alpha_1 = \cdots = \alpha_M = 0}}{\left(\dfrac{d^M}{d\alpha_1 \cdots d\alpha_M} J(\alpha, 0)\right)_{\alpha_1 = \cdots = \alpha_M = 0}}. \tag{49}$$

The derivatives in (49) can be approximated by classical finite differences.

For real valued functions $U : \mathbb{R}^M \to \mathbb{R}$, the (symmetric) finite difference approximation to the cross-derivative of U at the origin $(0, \ldots, 0) \in \mathbb{R}^M$ is

$$\frac{d^m U}{dx_1 \cdots dx_M}(0,\ldots,0)$$

$$\cong \frac{1}{\varepsilon_1 \cdots \varepsilon_M 2^M} \sum_{\sigma_1,\ldots,\sigma_M=0}^{1} (-1)^{\sigma_1+\cdots\sigma_M} U((-1)^{\sigma_1}\varepsilon_1,\ldots,(-1)^{\sigma_M}\varepsilon_M), \quad (50)$$

where the increments ε_j are suitably "small." In words, the sum is over all 2^M combinations of signs $\{+1,-1\}$. The constant $\varepsilon_1 \cdots \varepsilon_M 2^M$ in (50) cancels out of the ratio (49), leaving only the sums in numerator and denominator. The alternating signs in the sum (50) can lead to underflow for sufficiently small increments. Underflow can be reduced by accumulating the positive terms and negative terms separately and then taking the difference.

The number of terms in the finite difference form (50) is not impractical for small values of M. For values up to, say, $M = 10$, the difficulties encountered can be mitigated by fast multi-core computers. Whatever the limiting value of M, it is ultimately necessary to restrict the measurement space to one or more "windows" that contain at most M measurements.

6.2. Method 2: Maclaurin Series Expansion

PGFLs and their secular functions encode combinatorial information. Consequently, truncating any series approximation to them can be equivalent to a combinatorial constraint. Truncating the Maclaurin series after the linear term is shown to be such a case this subsection.

The secular functions in the examples of Section V have the form, for some choice of constant s and vector $\mathbf{c} \in \mathbb{R}^M$,

$$J(\boldsymbol{\alpha},\beta_1;\mathbf{y},x_1) = c_0 \exp(\mathbf{c}^T\boldsymbol{\alpha} + \pi(\boldsymbol{\alpha}) + \beta_1\pi(\boldsymbol{\alpha};x_1)), \quad (51)$$

where the function $\pi : \mathbb{R}^M \times S \to \mathbb{R}$ and $\pi(\boldsymbol{\alpha}) \equiv \int_S \pi(\boldsymbol{\alpha};s)\, ds$. Expanding the integrand $\pi(\boldsymbol{\alpha};x)$ in a Maclaurin series gives

$$\pi(\boldsymbol{\alpha};s) \cong \pi(\mathbf{0};s) + [\nabla\pi(\mathbf{0};s)]^T\boldsymbol{\alpha} + \tfrac{1}{2}\boldsymbol{\alpha}^T[\nabla^2\pi(\mathbf{0};s)]\boldsymbol{\alpha} + \cdots, \quad (52)$$

where $\nabla\pi(\mathbf{0};s) \equiv (\nabla\pi_\ell(\mathbf{0};s) : \ell = 1,\ldots,M) \in \mathbb{R}^M$ and $\nabla^2\pi(\mathbf{0};s) \in \mathbb{R}^M \times \mathbb{R}^M$ are the gradient and Hessian matrix of $\pi(\boldsymbol{\alpha};s)$, respectively, evaluated at $\boldsymbol{\alpha} = \mathbf{0} \in \mathbb{R}^M$. Substituting (52) into (51) and retaining only linear terms gives the approximate secular function

$$J(\boldsymbol{\alpha},\beta_1;\mathbf{y},x_1) \cong c_0 \exp\left(\mathbf{c}^T\boldsymbol{\alpha} + \int_S [\nabla\pi(\mathbf{0};s)]^T\boldsymbol{\alpha}\, ds + \beta_1[\pi(\mathbf{0};x_1) + [\nabla\pi(\mathbf{0};x_1)]^T\boldsymbol{\alpha}]\right). \quad (53)$$

It follows from (30) that the intensity function is

$$m_{[1]}(x_1) = \pi(\mathbf{0};x_1) + \sum_{\ell=1}^{M} \frac{\nabla\pi_\ell(\mathbf{0};x_1)}{\mathbf{c}_\ell + \int_S \nabla\pi_\ell(\mathbf{0};s)ds}. \quad (54)$$

A close examination of the expression (54) for, say, Example 3, shows that truncating the Maclaurin series to linear terms is tantamount to the combinatorial restriction that a target generate at most one measurement in at most one sensor. This constraint is not realistic in many problems.

Truncating the Maclaurin series after the quadratic or higher order term will result in different combinatorial restrictions, the nature of which is not studied here. The derivatives of these higher order expansions can be evaluated numerically for use in particle filter implementations to evaluate performance.

7. CONCLUSIONS

It is shown that functional derivatives of the PGFL are equivalent to ordinary derivatives of secular functions. Symbolic derivatives of secular functions can be found with widely available software. The secular function technique yields exact, not approximate, values of the functional derivatives of the PGFL. It lends itself to particle filter implementations because particle weights can be found by evaluating derivatives of the secular function, not functional derivatives of the PGFL. Embedding symbolic differentiation software in production code is somewhat unorthodox, but well within modern computing capabilities.

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Roy Streit is currently a Senior Scientist at Metron in Reston, Virginia. His research interests include multi-target tracking, multi-sensor data fusion, distributed systems, medical imaging, and signal processing, as well as statistical methods for pharmacovigilance and business analytics. Prior to joining Metron in 2005, he was in the Senior Executive Service at the Naval Undersea Warfare Center in Newport, RI, working primarily on the development, evaluation and application of multisensor data fusion algorithms in support of submarine sonar and combat control automation. He was an Exchange Scientist at the Defence Science and Technology Organisation (DSTO) in Adelaide, Australia, from 1987–1989. In 1999, he received the Solberg Award from the American Society of Naval Engineers for contributions to naval engineering through personal research, and in 2001 the Department of the Navy Superior Civilian Achievement Award. From 1996 to 2005 he served on the Sonar Technology Panel (Panel 9) of The Technical Cooperation Program (TTCP), a multinational governmental organization supporting scientific information exchange between member nations. He was President of the International Society for Information Fusion (ISIF) in 2012, and he continues to serve on the ISIF Board of Directors.

Dr. Streit is a Fellow of the IEEE. He is the author of a book entitled *Poisson Point Processes: Imaging, Tracking, and Sensing* published by Springer in 2010. It was translated into Chinese and published by Science Press, Beijing, in 2013. He has published papers in over a dozen refereed technical journals, and given numerous invited and contributed papers at international conferences and workshops. He holds nine U.S. patents. He is a Professor (Adjunct) in the Department of Electrical and Computer Engineering at the University of Massachusetts–Dartmouth. He received a Ph.D. in Mathematics in 1978 from the University of Rhode Island. He was a Visiting Scientist at Yale University from 1982–1984, and a Visiting Scholar at Stanford University from 1981–1982. He is currently an Affiliate Consultant with SUCCEED Educational Consultancy based in Rhode Island.